Mortality Experience in Asia-Pacific and Modelling and Management of Longevity Risk

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# Table of Contents

1 **Mortality Statistics**  
1.1 Increasing life expectancy  
1.2 Data sources and quality  
1.3 Population experience  
1.4 Industry experience  
1.5 Mortality projection results  

2 **Modelling of Mortality Improvement**  
2.1 The Lee-Carter method  
2.2 Poisson Lee-Carter model  
2.3 Non-linearity of the mortality index  
2.4 Uncertainty of mortality projections  
2.5 Projection of mortality curves  
2.6 Other projection methods  
2.7 Extrapolative and non-extrapolative approaches  

3 **Management of Longevity Risk**  
3.1 The longevity insurance market  
3.2 Demand and supply constraints  
3.3 More variety in product design  
3.4 Development of the life market  
3.5 Government initiatives  
3.6 Natural hedging  
3.7 Pricing methodologies  
3.8 The market conditions in Singapore  

4 **Mortality Indices**  
4.1 Research background  
4.2 CBD mortality indices  
4.3 Vector time series modelling  

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2012 Insurance Risk and Finance Research Centre
4.4 A graphical risk metric  
4.5 Concluding remarks

5 Multivariate Risk-Neutral Pricing Method  
5.1 Research background  
5.2 Bayesian risk-neutral method  
5.3 Reverse mortgages  
5.4 Bayesian pricing of $I_t$  
5.5 Bayesian pricing of the non-recourse provision  
5.6 Pricing reverse mortgages in Japan  
5.7 Concluding remarks

6 References

7 Acknowledgements
1 Mortality Statistics

In this section, we examine the past trends in life expectancy and mortality improvement for a number of areas in Asia-Pacific. We attempt to cover both population data and industry data, which are collected mainly from official agencies, actuarial bodies, and academic research work.

The phenomenon of rising life expectancy is briefly discussed in Section 1.1. The data sources and limitations are listed in Section 1.2. An analysis of population experience is given in Section 1.3. A study of industry experience is provided in Section 1.4. Some results of mortality projection are set forth in Section 1.5.
1.1 Increasing life expectancy

Human life expectancy has risen significantly since the 20th century until now. In 1900, life expectancy of Sweden was 52 years, one of the highest in that period. As of 2009, Japan is the world leader in life expectancy, which is 83 years. This upward trend is rather persistent, and as shown in Section 1.3, there is no sign of it ending or reversing in the foreseeable future. It is generally believed that rising life expectancy is driven by improved nutrition, hygiene, medical care, living environment, education, and lifestyle (in particular, reduction in smoking prevalence), while there are certain offsetting effects such as obesity (e.g. diabetes), epidemics, and catastrophes. The major causes of death have shifted from infectious diseases to chronic diseases, and there has been a rectangularization of the survival curve. (As noted in Cheung et al. (2005), however, the rectangularization trend of some Western countries is gradually replaced by an almost parallel shift of the survival curve to the right.) Historically, mortality at young ages declined more rapidly; in recent decades, old-age mortality improvement has turned out to be more important.

There are two main schools of thought about the future trend of life expectancy. The ‘optimists’ believe that life expectancy will continue to increase. For example, Oeppen and Vaupel (2002) find that best-practice life expectancy has increased by 2.5 years per decade for a century and a half. On the other hand, the ‘pessimists’ believe that life expectancy has an upper limit of about 85 years. For example, Carnes and Olshansky (2007) note that an elimination of both cancer and heart disease is needed to reach this limit. As noted in Bongaarts (2006), the pessimists seem to have made a concession recently, saying that the limit is a mortality schedule that cannot be reduced further ‘in the absence of medical interventions’. Carnes and Olshansky (2007) call themselves ‘realists’ instead and believe that life expectancy is ‘unlikely’ to exceed an average of around 85 years ‘in the absence of radical advances in the control of the ageing process’. In medical or biological views, while there are certain biological limits such as the Hayflick limit (the maximum number of times most human body cells can divide), the results of recent experiments using evolutionary and population genetic arguments oppose the existence of an upper limit to the lifespan (Tuljapurkar and Boe (1998)).

For product design and pricing and reserving purposes, it is important to consider these different views and study the past trends in detail. In the following, we will examine population and industry mortality experience for some regions in Asia-Pacific.
1.2 Data sources and quality

Table 1.1 below lists the sources of the data we use. Population data were collected by official agencies. These data are by age (single age or 5-year age group) and by sex, and are of good quality in general. For each population (except China) under consideration, there are 30 or more years of data. The experiences, however, are scarce at very high ages (say, ages 90 and above), and the corresponding death rates are rather volatile.

Table 1.1 Data sources

<table>
<thead>
<tr>
<th>Area</th>
<th>Population Data</th>
<th>Industry Data</th>
</tr>
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<tbody>
<tr>
<td>Taiwan</td>
<td>Human Mortality Database 1970-2010</td>
<td>-</td>
</tr>
</tbody>
</table>

For industry experience, the data were collected from major insurers and compiled by actuarial and insurance organisations. These data are by age (single age or 5-year age group) and by sex, and are generally based on life benefits, standard lives, ultimate experience, aggregating medical and non-medical policies, and combining smokers with non-smokers. These studies are periodic and the data are only available every couple of years. The main data age range is 20 to 65, and the data are very scarce at young and old ages. The data volumes are much less than those of more advanced markets such as the UK and the US, and there are often changes in participants, underwriting procedures, risk classification, target markets, and distribution methods between studies. (Moreover, the annuitant data, where available, are not used here because they are too scanty.)

Under the limitations as stated above, we use population data as the main basis for analysis. We then compare industry experience with population experience and interpret the industry trends with caution. We also briefly review some overseas experience on insured and annuitant lives for comparison purposes.
1.3 Population experience

Figure 1.1 below shows that the increasing trend of (period) life expectancy at birth has been prominent and persistent for the past couple of decades in Asia-Pacific. The differences between populations tend to decrease over time, particularly for males. Although earlier data are not available for some areas, it can be seen that generally (period) life expectancy at age 65 has started to increase significantly since the 1970s, highlighting the growing importance of old-age mortality improvement. The average life expectancy at birth of these populations (excluding China) is 84 years for females and 79 years for males in 2009, having increased from 77 and 71 respectively in 1980. The corresponding average life expectancy at age 65 is 22 years for females and 18 years for males in 2009, compared to 17 and 14 in 1980. Figure 1.2 also demonstrates that mortality decline occurs at all age ranges, with old-age improvement becoming more significant in recent decades. Figure 1.3 plots the annual rates of improvement in death rates for the last three decades. On average, the annual improvement rates are around 3% to 4% for ages 0 to 19, 2% for ages 20 to 84, and 1% for ages 85 and above.

Figure 1.1 Life expectancy at birth and at age 65
Figure 1.2a  Death rates (log scale)
Figure 1.2b  Death rates (log scale)

Figure 1.3a  Annual rates of improvement in death rates
Figure 1.3b  Annual rates of improvement in death rates

Figure showing annual rates of improvement in death rates for different age groups (20-39, 40-64, 65-84, and 85+) and genders (Females, Males) across different periods (1980-1990, 1990-2000, 2000-2010). The figures display data for Australia, China, Hong Kong, Japan, Singapore, and Taiwan.
1.4 Industry experience

For life insured experience, Figure 1.4 shows the annual rates of improvement in death rates for ages 20 to 64, using data from periodic industry studies. While the data volumes are much smaller than the population ones and there are often changes in basis between studies, it seems that overall, industry rates are somewhat higher than population rates (roughly by 1% on average). This observation is in line with a recent study in the US by Purushotham, Valdez, and Wu (2011), which deduce a population rate of 1% for females and 1.5% for males, with an insured rate being 0.5% higher for both sexes, for the future 10 to 20 years. They also deduce an annuitant rate being 0.25% higher than the population rate.

Note that there are safety margins embedded in the insured death rates of China and Japan. Since the exact sizes of the margins are not known, the calculated improvement rates may not truly reflect the underlying values.

Figure 1.4a Annual rates of improvement in death rates
As mentioned earlier, each set of industry data covers only a few time periods, and there are often changes in basis between studies. The results of applying a projection method (as described below) to these data would then be unstable and unreliable. Alternatively, we will apply a projection method, the Lee and Carter (1992) method, to population data instead. The Lee-Carter method serves as a benchmark method in practice, and in the following, we will show that the projection results are reasonable in general – the mortality index for each region is highly linear and no significant volatility is observed. These results suggest that the Lee-Carter method is satisfactory for projecting future mortality in Asia-Pacific using population experience, the volume of which is much larger than that of industry data.
1.5 Mortality projection results

We now apply the Lee and Carter (1992) method, as described in Section 2.1, to the population data collected, and project future death rates and life expectancy. The basic structure \( \ln m_{x,t} = a_x + b_x k_t + \varepsilon_{x,t} \) is straightforward, in which \( m_{x,t} \) is the central death rate, \( a_x \) is the general mortality schedule, \( b_x \) is the sensitivity of the log death rate to changes in the mortality index \( k_t \), and \( \varepsilon_{x,t} \) is the error term. This method is well known for producing highly linear time series of the mortality index empirically for many developed countries, which allow one to simply extrapolate the linear trend to project future mortality. To strike a balance between using as much data as available and data relevance (e.g. old-age mortality decline in recent decades), and based on Booth, Maindonald, and Smith (2002), to ensure \( b_x \) is less variant and \( k_t \) is more linear over time, we use the data from year 1980 onwards (except for China with only four data points). In line with the usual convention of having the (maximum) length of the projection period roughly equal to that of the fitting period, we project the death rates from the most recently observed data points to year 2040. In addition, the death rates of ages 90 and above are rather volatile and hence we exclude them in our projections. We then use the Coale and Guo (1989) method to ‘close out’ the life tables.

The patterns of \( a_x \) are similar to those in Figure 1.2. Figure 1.5 demonstrates the typical shapes of \( b_x \), indicating that the highest sensitivity to general mortality decline occurs at the two ends of the whole age range. Figure 1.6 shows clearly that \( k_t \) is highly linear (blue) for various populations in Asia-Pacific, and the declining trend can readily be extrapolated into the future (red). We also adopt the simulation approach in Li, Lee, and Tuljapurkar (2004) to construct 95% prediction intervals (dotted red). Finally, Figures 1.7 and 1.8 plot the observed and projected (period) life expectancy at birth and at age 65. The average projected life expectancy at birth (excluding China) in 2040 is 90 years for females and 85 years for males. The average projected life expectancy at age 65 in 2040 is 27 years for females and 23 years for males.

Note that one may simply increase the size of \( b_x(k_t - k_{t-1}) \) to allow for possibly higher improvement rates for insured and annuitant lives.
Figure 1.5a  The computed values of $b_x$
Figure 1.5b  The computed values of $b_x$

![Graphs showing computed values of $b_x$ for Singapore (Females) and (Males), and Taiwan (Females) and (Males).](image)

Figure 1.6a  The computed values of $k_t$

![Graphs showing computed values of $k_t$ for Australia (Females) and (Males), and China (Females) and (Males).](image)
Figure 1.6b  The computed values of $k_t$
Figure 1.7a  Life expectancy at birth

- **Australia (Females)**
- **Australia (Males)**
- **China (Females)**
- **China (Males)**
- **Hong Kong (Females)**
- **Hong Kong (Males)**
- **Japan (Females)**
- **Japan (Males)**
Figure 1.7b  Life expectancy at birth

Figure 1.8a  Life expectancy at age 65
Figure 1.8b  Life expectancy at age 65
2 Modelling of Mortality Improvement

In this section, we provide an overview of various existing methods for estimating future death rates and life expectancy. These methods have different characteristics and are suitable for different populations, age ranges, time periods, and types of data.

The actuarial and insurance literature hitherto has focused on extrapolative methods rather than non-extrapolative methods. The former provides an objective analysis using historical patterns while avoids handling the complicated relationships between mortality and economic, social, and environmental factors. In practice, the distinction is less clear-cut and a mix of different methods can be used. Broadly speaking, there are two main types of extrapolative methods, the so-called ‘principal components’ methods which are covered in Sections 2.1 to 2.4, and the ‘curve fitting’ methods as described in Section 2.5. Some more extrapolative methods are listed in Section 2.6. The relative merits of extrapolative and non-extrapolative methods are discussed in Section 2.7. Parts of this section are extracted from Li (2012d).
2.1 The Lee-Carter method

This method is proposed by Lee and Carter (1992) and it remains the most popular method for projecting future death rates. The basic structure is:

\[ \ln m_{x,t} = a_x + b_x k_t + \varepsilon_{x,t} \]  \hspace{1cm} (1)

in which \( m_{x,t} \) is the central death rate at age \( x \) in year \( t \), \( a_x \) describes the overall mortality schedule across age, \( b_x \) measures the sensitivity of the log death rate to changes in the mortality index \( k_t \), and \( \varepsilon_{x,t} \) is the homoscedastic error term with mean zero and variance \( \sigma^2 \). The \( a_x \)'s are computed from averaging \( \ln m_{x,t} \) over time. The \( b_x \)'s and \( k_t \)'s are then obtained by applying singular value decomposition (SVD) to \( \{ \ln m_{x,t} - a_x \} \), under two constraints \( \sum b_x = 1 \) and \( \sum k_t = 0 \). Finally, the \( k_t \)'s are re-calculated so that the fitted number of deaths and the actual number of deaths are the same for each year of data.

The strengths of this method are mainly its plain structure and its ability to produce a highly linear time series of \( k_t \) for many countries’ mortality data. This high linearity greatly facilitates projection of death rates and allows one to simply use a random walk with drift to model the \( k_t \) series:

\[ k_t = \mu + k_{t-1} + e_t \] \hspace{1cm} (2)

where \( \mu \) is the drift term and \( e_t \)'s are independent, identically distributed (iid) error terms having mean zero and variance \( \sigma^2_k \). The estimated drift term \( \hat{\mu} \) is usually negative and represents the broad mortality improvement across time. Future mean values are estimated as \( \hat{k}_t = \hat{\mu} + \hat{k}_{t-1} \) and \( \hat{m}_{x,t} = \exp \left( \hat{a}_x + \hat{b}_x \hat{k}_t \right) \). Moreover, to generate random samples of future death rates and construct prediction intervals, one may further assume \( \varepsilon_t \) and \( e_t \) are normally distributed and use (1) and (2) with the estimated parameters to perform simulation.

Some possible extensions of the Lee-Carter method are listed below:

(i) Carter and Lee (1992) model each sex separately, and also model both sexes jointly using a common \( k_t \) but two different sets of \( a_x \) and \( b_x \).

(ii) Lee (2000) suggest: (a) separate or joint projection of both sexes; (b) separate or joint projection of sub-populations; (c) separate projection of each cause of death; (d) setting lower limits for projected death rates; (e) using the most recent observed death
rates as the starting point of projection; (f) adding higher order terms; (g) permitting $\mu$ to change over time; (h) taking leader countries’ projected mortality as a bound.

(iii) Lee and Miller (2001) propose: (a) using year 1950 as the starting year of the fitting period (so that $b_x$ is less variant); (b) matching the life expectancy (rather than the number of deaths) to revise $k_i$; (c) taking the most recent observed death rates as the starting point of projection; (d) adding higher order terms.

(iv) Booth, Maindonald, and Smith (2002) and Booth, Tickle, and Smith (2005) make use of the Poisson distribution to revise $k_i$, construct a statistical measure to select the optimal fitting period, which makes $b_x$ less variant and $k_i$ more linear over time, and incorporate a total of $n$ factors as:

$$\ln m_{x,t} = a_x + \sum_{j=1}^{n} b_{x,j} k_{t,j} + \epsilon_{x,t} .$$

(3)

Renshaw and Haberman (2003b) also consider such higher order terms.

(v) Li, Lee, and Tuljapurkar (2004) adjust the Lee-Carter method for the situations when there are merely a few years of data at uneven intervals via re-formulating (2) as:

$$k_{t_0} - k_{t_n} = \mu \left( t_n - t_0 \right) + e_{t_0+1} + e_{t_0+2} + ... + e_{t_n} ,$$

(4)

in which data are collected only at $t_0$, $t_1$, $t_2$, ..., $t_n$.

(vi) Li and Lee (2005) propose the augmented common factor model for handling sub-populations jointly as:

$$\ln m_{x,t,i} = a_{x,i} + B_{x,i} K_i + b_{x,i} k_{t,i} + \epsilon_{x,t,i} ,$$

(5)

where the subscript $i$ indicates the sub-population $i$, $B_{x,i} K_i$ represents the common factor for all sub-populations, and $b_{x,i} k_{t,i}$ represents the specific factor for the sub-population $i$. The common factor refers to the key long-term trend in mortality improvement of the entire population and the specific factor refers to the short-term disparity from this trend for each sub-population. The mortality index $K_i$ is assumed to follow a random walk with drift, and the time component $k_{t,i}$ is assumed to follow an autoregressive (AR) model.

(vii) Wang and Lu (2005) propose modelling the number of deaths with the binomial distribution so that the covariance matrix for the parameters can be computed:

$$D_{x,t} \sim \text{Bi} \left( E_{x,t}, m_{x,t} \right) ;$$
\[
\ln m_{s,t} = a_s + b_s k_t, \quad (6)
\]

in which \( D_{s,t} \) is the number of deaths and \( E_{s,t} \) is the known exposure. (Haberman and Renshaw (2011) also use the binomial distribution.)

(viii) De Jong and Tickle (2006) use a time series approach to design a new framework which integrates estimation and forecasting.

(ix) Koissi and Shapiro (2006) investigate a fuzzy formulation of the Lee-Carter structure, which does not have the homoscedasticity constraint of the original method.

(x) Hyndman and Ullah (2007) smooth the log death rates across age for each year of data and set a number of factors. They model the time components with univariate time series models. Booth et al. (2006) and Shang, Booth, and Hyndman (2011) compare this approach with the Lee-Carter method and some other extensions.

(xi) Koissi and Shapiro (2008) allow both \( a_s \) and \( b_s \) to vary over time as a linear function and a 4\(^{th}\) order polynomial respectively. They use the forecast content function to determine the horizon beyond which the past values of \( k_t \) are no longer relevant for predictions.

(xii) Russolillo, Giordano, and Haberman (2011) extend the Lee-Carter structure to a three-way structure so as to model two or more countries jointly as:

\[
\ln m_{s,t,i} = a_{s,i} + b_{s,i} \lambda_j + \epsilon_{s,t,i}, \quad (7)
\]

in which \( \lambda_j \) is the parameter set for the country \( i \).
2.2 Poisson Lee-Carter model

Brouhns, Denuit, and Vermunt (2002) suggest modelling the number of deaths directly as a Poisson variable rather than using homoscedastic error terms of the original Lee-Carter method. This is because the death rates at older ages are more unstable than those at younger ages. The model structure is:

\[ D_{x,t} \sim \text{Pn} \left( E_{x,t}, m_{x,t} \right); \]
\[ \ln m_{x,t} = a_x + b_x k_t, \quad (8) \]

in which \( D_{x,t} \) is the number of deaths and \( E_{x,t} \) is the known exposure. The Poisson distribution is a natural choice for a counting random variable, and it furnishes a rigorous statistical framework for analysing mortality data. The model parameters are estimated by maximum likelihood with an iterative updating scheme, and standard diagnostic procedures can be taken to assess the model fit.

Some possible extensions of the Poisson Lee-Carter model are listed below:

(i) Bray (2002) and Currie (2006) study the age-period-cohort model:
\[ \ln m_{x,t} = a_x + k_t + \gamma_{t-x}, \quad (9) \]
where \( \gamma_{t-x} \) corresponds to the cohort effect.

(ii) Renshaw and Haberman (2003b) use two factors to improve the model fit as:
\[ \ln m_{x,t} = a_x + b_{x,1} k_{t,1} + b_{x,2} k_{t,2}, \quad (10) \]
in which \( k_{t,1} \) and \( k_{t,2} \) are modelled with univariate autoregressive integrated moving average (ARIMA) models. (This paper also modifies the original Lee-Carter method in a similar way.)

(iii) Koissi, Shapiro, and Hognas (2006) compare three ways of estimating parameters, including SVD, weighted least squares, and maximum likelihood under the Poisson assumption.

(iv) Renshaw and Haberman (2006) and Haberman and Renshaw (2009) study the addition of the cohort effect as:
\[ \ln m_{x,t} = a_x + b_{x,1} k_t + b_{x,2} \gamma_{t-x}, \quad (11) \]
in which \( \gamma_{t-x} \) refers to the cohort effect and \( k_t \) and \( \gamma_{t-x} \) are modelled with univariate ARIMA models. (The first paper also covers the corresponding Gaussian error
structure.) (Haberman and Renshaw (2011) suggest setting $b_{x,2} = 1$ to improve convergence.)

(v) Delwarde, Denuit, and Ellers (2007) construct a penalised log likelihood (or penalised least squares for the original Lee-Carter method) for parameter estimation such that $b_x$'s are smoothed in the estimation procedure.

(vi) Cairns et al. (2009), Dowd et al. (2010a, 2012b), and Cairns et al. (2011b) compare a number of models in detail, including (8), (9), (11), and the Cairns, Blake, and Dowd (2006) model (15) discussed in Section 2.5 and its extensions. Cairns, Blake, and Dowd (2008), Cairns et al. (2009), and Cairns et al. (2011b) further suggest some criteria to decide whether a stochastic mortality model is a good model or not, including positive death rates, ease of implementation, parsimony, transparency, capability to simulate sample paths, addition of cohort effects, capability to involve a nontrivial correlation structure between ages, quality of fit, robustness, allowance for parameter error, biological reasonableness, and plausibility of uncertainty forecast. (Haberman and Renshaw (2011) also provide a comparison of different models but they use the binomial distribution.)

(vii) Li, Hardy, and Tan (2009) assume there exist a number of clusters in each age-period cell and incorporate gamma-distributed heterogeneity between these clusters to release the mean-variance equality assumption of the Poisson distribution. The number of deaths then follows the negative binomial distribution instead.

(viii) Plat (2009a) tries to gather the desirable properties of the above models and suggest the following:
\[
\ln m_{x,t} = a_x + k_{1,t} + k_{2,t} (\bar{x} - x) + k_{3,t} \max(\bar{x} - x, 0) + \gamma_{x-t} ,
\]
where $\bar{x}$ is the mean of the age range, $k_{1,t}, k_{2,t},$ and $k_{3,t}$ are assumed to follow multivariate ARIMA models, and $\gamma_{x-t}$ follows an univariate ARIMA model.

(ix) Li (2012a) modifies the idea in Li and Lee (2005) as mentioned above and project the mortality of both sexes jointly as:
\[
\ln m_{x,t,i} = a_{x,i} + B_i + \sum_{j=1}^{n} b_{x,i,j} k_{i,j} ,
\]
where there is one common factor for both sexes and there are $n$ specific factors for each sex $i$. 
2.3 Non-linearity of the mortality index

Although the mortality index \( k_t \) of the Lee-Carter structure is shown to be highly linear for many countries’ data (e.g. Lee and Miller (2001)), a number of authors find evidence of some extent of non-linearity or some sort of structural change in the mortality index. For example, Booth, Maindonald, and Smith (2002) design a statistical measure to determine the optimal starting year of the fitting period so as to exclude the effect of the structural change. Renshaw and Haberman (2003a), Li and Chan (2007), Li (2010b), Coelho and Nunes (2011), Li, Chan, and Cheung (2011), and Li (2012a) also identify some structural changes in the mortality index. In particular, Li and Chan (2007) employ a systematic outlier detection process for outliers in the mortality index, and Li, Chan, and Cheung (2011) use Zivot and Andrews’ procedure to determine if there are any structural changes. Recently, Chan, Li, and Li (2012) investigate how the mortality indices of the model (15) can be jointly modelled with a multivariate time series model, which allows for certain non-linearity and cross-correlations (see Section 4).

As discussed in Li (2012a), the linearity of the mortality index (first time component) of the Lee-Carter structure is an important property for projecting mortality improvement, and it makes the projection much more straightforward and sensible than otherwise. Practically speaking, it would be reasonable to discard the irrelevant part of the past patterns and focus on the more recent developments for projection into the future.
2.4 Uncertainty of mortality projections

Besides using the original Lee-Carter method and assuming normal error terms to perform simulation as noted above (which includes only process error but not parameter error), there are several different ways to allow for uncertainty in mortality projections. One approach is Bayesian modelling and Markov chain Monte Carlo (MCMC) simulation.

As noted in Li (2012b), under the Bayesian framework, the main step is to obtain the posterior distribution of the unknown parameters and quantities given the data set. The posterior density function is deduced from $f(\theta|D) \propto f(D|\theta)f(\theta)$, where $f$ represents a density function, $\theta$ denotes the unknown parameters and quantities, and $D$ means the data. Owing to the complication of the mortality models, it is problematic to derive an explicit expression for the posterior density. One practical alternative is to use MCMC simulation, in which random samples are generated from a Markov chain having a stationary distribution equivalent to the posterior distribution. Different inferences can then be drawn from the sampled distribution. One may develop his or her own MCMC algorithm or use WinBUGS (Spiegelhalter et al. (2003)) to perform MCMC simulation.

As discussed in Li (2012b), there are some clear advantages of Bayesian modelling in mortality projection. Firstly, prior information can be formally incorporated into the modelling process. Secondly, both the Lee-Carter structure and the random walk with drift can be handled simultaneously within the same framework, in contrast to the rather incoherent two-step estimation procedure of the original Lee-Carter method. Moreover, the MCMC simulation technique offers a convenient means to sample distributions and construct probability intervals for the parameter estimates and model forecasts, in which both process error and parameter error are taken into account.

Some previous studies on Bayesian mortality modelling include Czado, Delwarde, and Denuit (2005), which implement a Poisson Lee-Carter model with a deterministic trend model for the mortality index. Pedroza (2006) formulates the Lee-Carter method into a state-space model, with Gaussian errors and a random walk with drift for the mortality index. Koissi and Shapiro (2006) propose a fuzzy formulation of the Lee-Carter method. Reichmuth and Sarferaz (2008) incorporate macroeconomic covariates into the Lee-Carter structure with Gaussian errors. Kogure, Kitsukawa, and Kurachi (2009) test both the Gaussian and Poisson formulations and compare three different model structures (random walk with drift,
deterministic trend model, and stationary model) for the mortality index. Cairns et al. (2011a) integrate the age-period-cohort model into a Bayesian framework designed for co-modelling a large population and a small sub-population. Li (2012b) investigates the use of Bayesian analysis for populations with only a few years of data.

For example, based on Kogure, Kitsukawa, and Kurachi (2009), the Gaussian and Poisson error structures and the prior distributions can be formulated as:

\[
\begin{align*}
\ln m_{x,t} &= a_x + b_x k_t + \varepsilon_{x,t}, \\
D_{x,t} &\sim \text{Pn}\left(E_{x,t} \exp\left(a_x + b_x k_t\right)\right), \\
k_t &= \mu + k_{t-1} + e_t, \\
a_x &\sim \mathcal{N}(0, \sigma_a^2), \\
b_x &\sim \mathcal{N}\left(\frac{1}{\text{no of age groups}}, \sigma_b^2\right), \\
\mu &\sim \mathcal{N}(\mu_0, \sigma_\mu^2), \\
\varepsilon_{x,t} &\sim \mathcal{N}(0, \sigma_\varepsilon^2), \\
e_t &\sim \mathcal{N}(0, \sigma_e^2), \\
\sigma_\varepsilon^{-2} &\sim \text{Gamma}(\alpha_\varepsilon, \beta_\varepsilon), \\
\sigma_k^{-2} &\sim \text{Gamma}(\alpha_k, \beta_k),
\end{align*}
\]

where \(m_{x,t}\), \(D_{x,t}\), and \(E_{x,t}\) are the central death rate, number of deaths, and known exposure, \(\sigma_a^2\), \(\sigma_b^2\), and \(\sigma_\mu^2\) are the variances of the prior distributions, \(\mu_0\) is the mean of the drift term \(\mu\), and \(\alpha_\varepsilon, \beta_\varepsilon, \alpha_k, \beta_k\) are the parameters of the prior distributions of the error terms’ inverse variances.

There are other ways to assess the level of uncertainty. For example, Olivieri (2001) considers scenario-based projection. (As stated in Crawford, De Haan, and Runchev (2008), most insurers in the UK and the US currently use rather basic methods such as stress tests for mortality modelling, while more advanced stochastic modelling is mainly used for economic and market risks.) Li, Lee, and Tuljapurkar (2004) take into account the parameter error of the drift term by using its standard error and the normal distribution. Koissi, Shapiro, and Hognas (2006) use the residual bootstrap to construct probability intervals. Renshaw and Haberman (2008) then compare three simulation strategies, in which both process error and parameter error can be incorporated:
(i) Semi-parametric bootstrap – simulate pseudo data (number of deaths) from the fitted Poisson distribution and then obtain samples of parameter estimates from fitting the model again to these pseudo data.

(ii) Parametric Monte Carlo – simulate samples of parameter estimates directly from a multivariate normal distribution.

(iii) Residual bootstrap – resample residuals with replacement and generate pseudo data (number of deaths), and then obtain samples of parameter estimates from fitting the model again to these pseudo data.

As discussed in Pitacco (2004), all of process error (random fluctuations), parameter error (uncertainty in parameter estimation), and also model error (model misspecification) should be addressed properly.
2.5 Projection of mortality curves

The Lee-Carter method is arguably the most influential method for mortality projection amongst academics and practitioners. This method is adopted by the Social Security Administration (SSA) in the US, the Continuous Mortality Investigation (CMI) in the UK, and the United Nations. Although it is widely used, it has a plain structure, and it allows various extensions, this method may produce certain irregularities in the projected death rates, and the projected age profile may depart significantly from historical patterns, as commented in McNown (1992).

McNown and Rogers (1989) propose a different method, in which the Heligman and Pollard (1980) mortality curve is fitted to each year of data for the entire age range to 85. Univariate ARIMA models are then applied to the computed time series of the eight parameters to project future mortality. Thus, the whole age profile is taken into account, and substantial deviance from historical patterns could be avoided in the projection. Moreover, this method also enables one to compare the mortality profiles across time and between populations. Along the same line, McNown and Rogers (1992) project three parameters of the multiexponential model and fix the remaining six parameters to forecast cause-specific mortality. Congdon (1993) also sets forth the idea of projecting the parameters of a ‘reduced’ version of the Heligman-Pollard curve (by fixing some parameters or considering only the old-age part) or a relational approach (with four parameters). Cairns, Blake, and Dowd (2006) advocate a two-factor stochastic model for the age range of 60 to 90 and project the two parameters with a bivariate random walk with drift (which is the same as projecting the reduced Heligman-Pollard form proposed by Congdon (1993)):

\[
\text{logit } q_{x,t} = k_{t,1} + k_{t,2} x \quad \text{or} \quad \text{logit } q_{x,t} = k_{t,1} + k_{t,2} (x - \bar{x}) ,
\]

in which \( q_{x,t} \) is the mortality rate, the probability of death within a year, at age \( x \) in year \( t \) and the structure is identical to the old-age part of the Heligman-Pollard curve. (Haberman and Renshaw (2011) also use the logit of the mortality rates when comparing different models.) Cairns et al. (2009) further incorporate additional period and cohort terms into the model structure and test the extended models on the age range of 60 to 89. Gaille and Sherris (2010) project the parameters of the entire Heligman-Pollard curve using a vector error correction model (VECM). On the other hand, Li (2012c) project the parameters of the logistic model proposed by Thatcher (1999) with a bivariate random walk with drift for the age range of 60
to extreme ages:

\[
\logit \mu_{x,t} = k_{t,1} + k_{t,2} x ,
\]  

(16)

where \( \mu_{x,t} \) is the force of mortality at age \( x \) in year \( t \). According to Thatcher (1999), the logistic model is shown to be clearly closer to a large set of old-age data than a number of widely used models including the Gompertz, Weibull, and Heligman-Pollard models. This model also produces satisfactory results even when fitted to ages below 100 and then extended to extreme ages. Bongaarts (2005) also modifies the logistic model into the shifting logistic model and extrapolates its two time-varying parameters to project mortality for the age range of 25 to 109.

As noted in Li (2012c), there are two problems behind this approach of projecting the parameters of a mortality curve for the entire age range. First, estimation of numerous parameters, e.g. eight parameters in the Heligman-Pollard curve, for every year is problematic and convergence of results may not occur. Second, there is a dilemma between using univariate and multivariate time series models: the former is easier to implement but completely ignores the relationships between the parameters; the latter takes the interrelations into account but can be much more complex to handle. If the concern is chiefly on older ages, e.g. for annuity providers, these problems can be lessened by using the idea in Congdon (1993), in which only the old-age element is involved. The number of parameters is then smaller, which eases parameter estimation and permits one to use multivariate models with lower dimensions, e.g. Cairns, Blake, and Dowd (2006) and Li (2012c).
2.6 Other projection methods

Some other projection methods are listed below:

(i) Bell (1997) compares applying the random walk with drift directly to the log death rate of each age against the Lee-Carter method and the McNown and Rogers (1989) approach.

(ii) Sithole, Haberman, and Verrall (2000) and Renshaw and Haberman (2003a, 2003c) use generalized linear modelling (GLM) and mortality reduction factors to project future death rates. This approach treats time as a known covariate, while the Lee-Carter method treats the period effect as a factor.


(v) Denton, Feaver, and Spencer (2005) apply time series models to the first differences of log death rates and allow for correlations between different age-sex groups.


(vii) Neves and Migon (2007) apply dynamic models under the Bayesian context to model the evolution of mortality over time.

(viii) Babel, Bomsdorf, and Schmidt (2008) utilise a panel data model for the rate of change in death rates and distinguish between a common time effect and an age-specific effect.

(ix) Girosi and King (2008) establish a sophisticated Bayesian framework to incorporate covariates and prior information in order to improve mortality forecasts.

(x) Hari et al. (2008a, 2008b) model the log death rates directly by the random walk with drift and allow for time-varying trends.

(xi) Gao and Hu (2009) use the generalized dynamic factor model (GDFM) and the multivariate generalized autoregressive conditional heteroscedasticity (GARCH) model to describe the conditional heteroscedasticity of mortality.

(xii) Hatzopoulos and Haberman (2009, 2011) model the death rates with orthonormal polynomials via GLM and obtain the principal components of the fitted parameters.
They then apply dynamic linear regression (DLR) models to these principal components to project mortality.

(xiii) Plat (2009b) constructs a stochastic model for the relationship between portfolio mortality and population mortality.

(xiv) Yang, Yue, and Huang (2010) use the first two principal components of the log death rates to project mortality.

Our discussion so far has focused mainly on discrete-time models, in which the death rates are modelled on an annual basis. Cairns, Blake, and Dowd (2008) review a number of continuous-time models, which make use of stochastic processes and would be useful for derivative pricing.
2.7 Extrapolative and non-extrapolative approaches

The methods stated hitherto are extrapolative in nature. Booth (2006) and Booth and Tickle (2008) provide a broad review of demographic forecasting methodologies. They indicate that there are generally three approaches: extrapolation, expectation, and explanation. The expectation approach involves the use of individuals’ expectations and experts’ opinions. This approach can cater for the possibility of future structural changes and unexpected events. As noted in Lee and Miller (2001), however, there has been historically a pessimistic bias of experts’ opinions, which suggests that present knowledge reflects on current limitations but not future means of overcoming them. This approach also tends to be limited to aggregate measures without more specific details. Alternatively, the explanation approach uses theories to explain the relationships between demographic quantities and underlying economic, social, and environmental factors. This approach is supposed to provide a fundamental solution to forecasting mortality. However, most relationships studied to date are only hypotheses, because of the complication of the problem and the lack of long-term, detailed data, and there is a high risk of misspecifying the model.

Extrapolation remains the most popular approach. The major assumption is that the future will be a continuation of the past, at least to some extent. Some authors criticise this approach being overly simple (e.g. Gutterman and Vanderhoof (1998)), in the sense that it merely assumes (some of) the past patterns would repeat in the future and does not incorporate experts’ opinions or the structural relationships between the demographic and other variables. Still, studying the past patterns and trends carefully forms a solid basis for understanding how mortality changes across time, and projecting these past trends into the future serves as a robust first-step or benchmark for further analysis. In practice, the division between different approaches is not that clear-cut and one can always modify the extrapolation approach by experts’ opinions or related variables where suitable. For insurers’ pricing and reserving purposes, this approach provides a transparent, systematic, and statistically sound means to estimate future mortality. Perhaps, the extrapolation approach is suitable for the short to medium term (say, 20 years or so), while experts’ opinions, informed judgement, comparison to other populations, and scenario testing are needed to supplement the analysis for the longer term. Li (2012d) provides detailed information on the existing extrapolation methods and discusses some possible developments of the methods.
3   Management of Longevity Risk

In this section, we review the concept and practice of longevity risk management. We attempt to identify the existing problems behind the longevity insurance market, and to explore feasible solutions for tackling the various underlying difficulties. We seek to understand the needs and constraints in the market from the perspectives of the consumers and providers of insurance products. We also cover some pricing methodologies for longevity risk.

A brief description of the longevity insurance market and its problems is given in Section 3.1. The underlying demand and supply constraints are discussed in Section 3.2. Some possible solutions are outlined in Sections 3.3 to 3.5, including more variety in product design, development of the life market, and government initiatives. The effect of natural hedging is explored in Section 3.6, and some pricing methods are covered in Section 3.7. Finally, the Singapore market conditions are discussed in Section 3.8.
3.1 The longevity insurance market

It is a global phenomenon that life expectancy continues to increase and there is no sign of reversal in the foreseeable future. This is particularly so for the Asia-Pacific region, as shown in Section 1. On one side, it represents huge success from the 20th century until now, in the areas of economic growth, education, living conditions, health care, and medical technology. On the other side, it raises serious concerns over the impact of longevity risk, which has different implications for different parties. For an individual, longevity risk is the risk of outliving his or her resources during retirement because of living longer than expected. For an insurer or a pension scheme, longevity risk is the risk of paying more than expected due to mortality improvement. Roughly speaking, there are two types of longevity risk. One is the non-systematic longevity risk, which arises from random fluctuations between individuals and can be diversified by increasing the portfolio size. The other is the systematic longevity risk, which affects all individuals coherently and cannot be diversified by pooling. For example, in terms of the Poisson Lee-Carter model, \( D_{s,t} \) (given the mean) involves the non-systematic risk and \( e_t \) represents the systematic risk. An individual probably tends to worry more about the non-systematic risk, while an insurer would be anxious about how to manage the systematic risk properly.

As mortality improvement continues and the ageing problem (in terms of financial provision) worsens, and as governments gradually reduce their support for retirement programs and retirement ages have not increased to accommodate rising life expectancy, there is a genuine and pressing need for a strong and well-developed longevity insurance market. According to Creighton et al. (2005), it has taken about 75 years for Western countries’ elderly (60+) proportion to increase from 7% to over 21%, while this kind of transition for China and India is forecast to take only around 25 years. Over the past few decades, there has also been a trend for governments and employers to transfer the responsibility of retirement provision to individuals via the implementation of defined contribution (DC) schemes, in which the members are required to save for their own retirement during their working lives (bearing investment risk). On retirement, these schemes often allow the members to take their retirement benefits as a lump sum or withdrawals for a certain period of time, without the requirement to buy an annuity (so exposed to longevity risk). For example, a number of countries such as Australia have been operating DC schemes...
for one or two decades. Japan has recently implemented DC schemes, and China and India have also introduced a DC-style framework. All these trends clearly indicate that millions of people around the world will need longevity solutions for several decades to come, as they try to find ways to make their savings last for their retirement years.

In theory, a life annuity, which provides a series of regular fixed payments until death, would be an ‘optimal’ solution for an individual to deal with his or her own longevity risk. There are also potentially many ways to design and structure an annuity to meet the specific needs of a customer. In practice, however, even the world’s largest annuity market, the UK, which has grown significantly in the recent years, is still rather small compared to the life insurance market in terms of premium income. All the longevity insurance markets are indeed very small, especially in Asia. Japan, despite being the world leader in life expectancy, has seen little development of longevity insurance products to date.

For example, in Singapore, the main source of financial support for 63% of those aged 65 and over in 2010 was from allowances given by children, while for 3% it was from allowances given by the spouse. For 12% the main source was income from employment or business, while for 11% it was from savings or interests earned. For 3% it was from rental, dividends, annuities, or trusts, and for 8% it was from other sources (Wong and Teo (2011)). There are only 72,285 annuity policies in force at the end of 2010 (MAS (2010)), though this number is expected to increase significantly when mandatory annuitization becomes effective in 2013 (see Section 3.8). In Hong Kong, of those aged 60 and over in 2000, 58% received financial support from their families, 12% received a salary from employment, and 15% received retirement benefits, mostly in the form of a lump sum (Chi (2004)). The number of annuity policies in force at the end of 2010 is 55,550, and this number has been declining gradually from 60,681 policies in 2002 (Office of the Commissioner of Insurance (2004, 2010)). In Taiwan, for those aged 65 and over in 2010, 24% of income was from employment, 10% was from entrepreneurial income, 24% was from property income and imputed rent income, and 42% was from transfer receipts (12% from government, 10% from social insurance, and 20% from individuals) (DGBAS (2010)). For older people in Malaysia in 2005, 38% of income was from family members (almost entirely from children), 27% was from wage income and business, 18% was from pensions, 9% was from rent, and 8% was from other sources (Ong and Hamid (2010)). These figures suggest that the incomes of the elderly in Asia-Pacific are largely exposed to longevity risk without suitable protection.
3.2 **Demand and supply constraints**

There exist some fundamental problems that hinder the growth of the much needed longevity insurance market. Below are some constraints on the demand side, most of which are stated in Creighton et al. (2005):

(i) **Adverse selection** – Those people who are healthy will be more likely to buy annuities while those who do not expect to live long will be much less interested. This difference will drive annuity prices up and cause a spiral effect.

(ii) **Bequest motives** – Some people may wish to bequeath their assets to the children and be reluctant to annuitize their wealth. This is a common norm, particularly within Asian cultures.

(iii) **Myopia and poor financial planning** – Some people simply may not understand the importance of proper long-term planning for their retirement.

(iv) **Lack of understanding of longevity risk and longevity products** – Many people tend to focus on the risk of dying too soon rather than living too long, and may not truly realise the benefits of holding a life annuity.

(v) **Lack of annuity choices** – Some may find it hard to identify reasonable annuity choices, as most markets are very thin and options are limited. Some people are concerned about inflation risk, but there are very few inflation-linked products available (overseas) and their prices are perceived as too high.

(vi) **Flexibility for possible future expenditure** – Some may prefer to keep the control of their assets as a precaution for unexpected future expenditure.

(vii) **Reluctance to plan rationally for the final years** – It is probably human nature that people often prefer to avoid or postpone thinking carefully about their final years.

(viii) **Distrust of insurers** – Some may find it difficult to accept the idea of transferring the control of a large part of their assets to a financial institution and worry about the financial health of the institution.

On the other hand, Creighton et al. (2005) also point out some constraints on the supply side:

(i) **Uncertainty of future performance** – Due to the long-term nature of life annuities, it is not a straightforward exercise for insurers to price such products properly and calculate the necessary reserves. There is much uncertainty over future mortality,
investment returns, interest rates, and inflation.

(ii) Limited appetite of reinsurers – Because of those uncertainties stated in the previous point, reinsurers are generally reluctant to provide cover for large life annuity pools.

(iii) Lack of creditworthy long-term bonds – Insurers and reinsurers need investment-grade long-term bonds to hedge the interest rate risk in their annuity portfolios. The current stock of government and corporate bonds, however, is far from sufficient to support the growth of the longevity insurance market. The existing bonds are also generally too short in duration for this purpose.

Some possible solutions to these problems include more variety in product design, development of the ‘life market’, and government initiatives. Some more advanced markets, such as the UK and the US, are moving in these directions.
3.3 More variety in product design

A life annuity removes both longevity risk and investment risk from an individual, as it provides a series of regular fixed payments until death. If it is inflation-linked, it further removes inflation risk. Because it then leaves no risk to the person, this ‘original’ form of life annuity can become very expensive. More flexible risk sharing between the consumers and providers is needed to make a product marketable to the public while still manageable for the insurer in terms of pricing and reserving. Some real-life examples overseas and other suggestions are given below, which are discussed in Creighton et al. (2005) and Crawford, De Haan, and Runchey (2008):

(i) Variable annuity – This kind of annuity is being sold in the UK. The payments are adjusted along with the investment performance of the underlying assets. It gives the annuitant some control over asset allocation. Investment risk is shared between the insurer and annuitants. Other possible additional features include setting a minimum on the investment return, term of payments, death benefit, withdrawal amount, or income level.

(In the US, variable annuities often refer to phased withdrawals instead, for which at certain time intervals retirees withdraw their capital within maximum and minimum limits based on life expectancy. In this way, the retirees may use up all or most of the money by the pre-specified life expectancy and those who live longer than expected may have little resources at the end.)

(ii) Group self-annuitization – Piggott, Valdez, and Detzel (2005) suggest adjusting the payments along with both the investment performance of the underlying assets and the overall mortality experience of the annuity pool. It leaves the systematic longevity risk to the annuitants, while the non-systematic longevity risk can be diversified by increasing the portfolio size by the insurer. A similar product, called the participating annuity, is being sold in the UK, in which the annuitants share in certain investment and mortality gains of the insurer.

(iii) Impaired life annuity – This kind of product is being sold in the UK. It is a niche market for those who have worse than average health. It is cheaper than a standard life annuity. (Underwriting factors other than health may also be explored, e.g. location is used in the UK.)
Reverse mortgage – There are small markets in the US, Australia, Canada, and the UK. This product allows a person to borrow against the value of his or her own home property and is typically a non-recourse loan. Those people who have their wealth tied up in their properties can use this kind of product to release their capital. The principal and interest are repaid when the borrower dies. The amount borrowed can be received in the form of an annuity, a lump sum, or a line-of credit (the last two of which are basically phased withdrawals). For the annuity form, the insurer bears both longevity risk and house price deflation risk.

In principle, a variety of features can be added to these products to make them more appealing to potential customers and to make it easier for the insurer to manage the underlying risks. At this stage, however, the longevity insurance markets worldwide are still under-developed and the choices of products are limited in general. It appears that development of a robust ‘life market’ and provision of suitable government incentives are necessary impetuses for more product innovations amongst insurers.
3.4 Development of the life market

The so-called ‘life market’ is a market where (standardised or over-the-counter) mortality or longevity linked securities or liabilities are traded. Insurers and reinsurers can then spread their risks amongst one another or with certain investors who want to diversify their portfolios with an uncorrelated market sector. Recently, there are a number of new developments in the UK and Europe. For example, new life companies have been set up to buy out the defined benefit (DB) pension liabilities of the employers in the UK. Some insights can be drawn from these experiences for the Asia-Pacific markets. Creighton et al. (2005), Cairns, Blake, and Dowd (2008), and Blake et al. (2011) present a list of examples as follows:

(i) 3-year Swiss Re mortality bond in 2003 – Reduction in principal repayment is triggered if the mortality index exceeds a pre-specified level. Mortality risk (from catastrophes) is transferred from the issuer to the bondholders. It was over-subscribed and there were further issues by Swiss Re in 2005 and 2007, by Scottish Re in 2006, and by AXA in 2006.

(ii) 25-year EIB / BNP Paribas survivor bond in 2004 – Coupon payments are linked to the number of survivors in a reference population. Longevity risk is transferred from the bondholders to the issuer. It was undersubscribed probably because of the price being too high, the concern over basis risk (arising from the difference between the population to be hedged and the reference population) and tail risk (arising from exposure after the term of 25 years), and also limited reinsurer appetite.

(iii) Swap between Swiss Re and Friends’ Provident, UK in 2007 – Swiss Re receives a fee for assuming responsibility for an annuity portfolio written by Friends’ Provident. Longevity risk is transferred from the latter to the former. It was constructed as an insurance contract but not a capital market instrument.

(iv) 5-year and 10-year Goldman Sachs survivor swap in 2007 – One party pays a series of fixed payments to another party who pays in return a series of payments linked to the number of survivors in a reference population. Longevity risk is transferred from the former to the latter. It is a standardised contract and the mortality index is called Qx.X.LS. There is basis risk for certain users. (The mortality index was designed for trading life settlements. It was closed down in 2009 due to insufficient commercial
activity and the controversy associated with life settlements.)

(v) JPMorgan $q$-forward in 2007 – A payment based on an initially agreed mortality rate is exchanged with a payment based on the realised mortality rate of a reference population on maturity. (JPMorgan also released the LifeMetrics Indices in the same year.)


(vii) Deutsche Boerse released the Xpect Age and Cohort Indices in 2008, with the corresponding survivor swaps being offered from 2009.

(viii) Credit Suisse, using its own longevity index, entered into a survivor swap with Centurion Fund Managers in 2008.

(ix) Since 2008, there are a number of capital market survivor swaps in the UK, in which the insurers and the pension funds transfer their longevity risk to certain investors, hedge funds, and reinsurers, arranged by some investment banks.

(x) The Life and Longevity Markets Association (LLMA) was established in 2010 in the UK to promote the development of a liquid life market. The members include AXA, Deutsche Bank, J. P. Morgan, Legal & General, Pension Corporation, RBS, and Swiss Re, later joined by Morgan Stanley, UBS, Aviva, and Munich Re.

(xi) 8-year Swiss Re insurance-linked securities (ILS) notes in 2010 – These notes are devised to hedge Swiss Re’s own longevity risk.

Cairns, Blake, and Dowd (2008) discuss the idea of a ‘longevity-linked security’ (LLS), which is similar to the concept of mortgage-backed securities. On one side, the insurers enter into survivor swaps (for their own longevity-linked cashflows) with a special purpose vehicle (SPV). On the other side, the investors purchase survivor bonds (based on a survivor index) from the SPV. Longevity risk is transferred from the insurers to the bondholders. The SPV bears the basis risk. These authors also mention the possibility of mortality or longevity linked derivatives such as options, caps, and floors.
3.5 Government initiatives

Governments play an important role in overcoming the various demand and supply constraints. Below are some possible directions, some of which are discussed in Creighton et al. (2005) and Crawford, De Haan, and Runche (2008):

(i) The retirement system has a significant impact on the development of the longevity insurance market. For example, countries like the UK with some form of mandatory annuitization, which reduces adverse selection and increases market depth, have more developed annuity markets. A larger pool of purchasers and more competition amongst insurers may lead to more diverse and innovative products. There can be different formats and levels of mandatory annuitization, such as stipulating a minimum proportion of assets to be annuitized, setting the annuity conversion rates, or simply making annuitization the default option. Table 3.1 lists the benefit types of the retirement schemes of some markets in Asia.

(ii) More promotion and better education can be provided to the general public, regarding the impact of longevity risk, the importance of early planning for retirement, and the benefits of holding longevity insurance products. Various tax incentives can also be used to encourage individuals to purchase longevity products.

(iii) Some form of government guarantees or sponsorships can be provided to encourage the offer of certain longevity insurance products. For example, the US reverse mortgage market has been encouraged by a government-sponsored program which provides lenders with mortgage insurance for the event that the loans exceed the property values.

(iv) More long-term bonds (say, 30 years and over) can be issued with which insurers and reinsurers can hedge their interest rate risk. In Singapore, the maximum duration of government bonds is 30 years. In Hong Kong, the maximum duration is only 10 years, and there is a recent introduction of 3-year inflation linked bonds called iBonds.

(v) Development of a healthy longevity insurance market and a robust life market can be supported via suitable government policies and tax incentives. As the longevity related problems continues to aggravate, it is important for governments to provide the right initiatives for the private sector in the search of longevity solutions.

(vi) Academic and industry research can be promoted through research grants or funding.
In particular, academic research has proven to be forward-thinking and plays a crucial role in exploring market solutions, including risk modelling, pricing methodologies, and hedging strategies.

(vii) More comprehensive collection and study of insured and annuitant mortality experience can be carried out, similar to the regular publications by the Continuous Mortality Investigation (CMI) in the UK. This would greatly enhance the industry’s knowledge of longevity risk and allow the industry practitioners to design, price, and hedge longevity products with more confidence.

(viii) More data on the expenditures of the elderly can be collected and perhaps a separate consumer price index (CPI) can be constructed with these data. This kind of information would be useful for consumers’ education and product development.

Table 3.1  Benefit types of different retirement schemes

<table>
<thead>
<tr>
<th>Area</th>
<th>Generally Used Benefit Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Australia</td>
<td>lump sum</td>
</tr>
<tr>
<td>China</td>
<td>annuity + lump sum</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>lump sum</td>
</tr>
<tr>
<td>India</td>
<td>annuity + lump sum</td>
</tr>
<tr>
<td>Indonesia</td>
<td>drawdown or lump sum</td>
</tr>
<tr>
<td>Malaysia</td>
<td>drawdown</td>
</tr>
<tr>
<td>Singapore</td>
<td>annuity + lump sum</td>
</tr>
<tr>
<td>Taiwan</td>
<td>annuity or lump sum</td>
</tr>
</tbody>
</table>

Source: Social Security Administration (2011)
3.6 Natural hedging

If mortality improves more than expected, a life insurer gains while an annuity writer makes losses. If mortality improves less than expected, the situation is reversed. This offsetting effect is called natural hedging – a ‘natural’ effect for an insurer who sells both life insurance and annuities. An insurer with its business more balanced between life policies and annuities is supposed to have more stable aggregate liability cashflows than otherwise. The existence of this hedging effect has long been recognised, but very little research has been done on its significance and how to utilise it in practice. For example, Cox and Lin (2007) find that insurers who have more balanced business tend to charge lower annuity prices. They also consider the impact of mortality improvement and mortality shock on a hypothetical insurer with different business compositions, and find that the total present value of liabilities has lower variability when the business mix is more balanced. They further propose a mortality swap between a life insurer and an annuity writer to create a natural hedge.

In the following, we use insured and annuitant data from the CMI Reports 17 and 23, and perform some stress testing to analyse the potential benefits of natural hedging. Firstly, we construct a hypothetical insurer with the business mix in Table 3.2 below:

### Table 3.2 Business mix

<table>
<thead>
<tr>
<th>Age</th>
<th>Permanent Assurances</th>
<th></th>
<th>Immediate Annuities</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Death Benefits ($m)</td>
<td></td>
<td>Annual Payments ($m)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Males</td>
<td>Females</td>
<td>Males</td>
<td>Females</td>
</tr>
<tr>
<td>20</td>
<td>150</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>25</td>
<td>300</td>
<td>200</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>450</td>
<td>300</td>
<td>0</td>
<td>0</td>
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<tr>
<td>35</td>
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<td>400</td>
<td>0</td>
<td>0</td>
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<td>45</td>
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</tr>
<tr>
<td>60</td>
<td>30</td>
<td>20</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>65</td>
<td>30</td>
<td>20</td>
<td>42</td>
<td>28</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>75</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>80</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Total</td>
<td>3,000</td>
<td>2,000</td>
<td>120</td>
<td>80</td>
</tr>
</tbody>
</table>
Based on the data collected, Table 3.3 shows our assumption of the expected annual rates of mortality improvement in the future:

**Table 3.3 Expected annual rates of mortality improvement**

<table>
<thead>
<tr>
<th>Age Range</th>
<th>Permanent Assurances</th>
<th>Immediate Annuities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Males</td>
<td>Females</td>
</tr>
<tr>
<td>17-60</td>
<td>0.5%</td>
<td>0.5%</td>
</tr>
<tr>
<td>61-120</td>
<td>1.0%</td>
<td>1.0%</td>
</tr>
</tbody>
</table>

We then consider a number of unexpected (rather extreme) mortality movements and examine the corresponding impacts on the present value of liabilities for two different cases – having a business mix as in Table 3.2 or writing life policies (or annuities) only. The increases in the present value of liabilities can be seen as additional capital requirements to cover certain unexpected (extreme) events. The interest rate is assumed to be 3% p.a. Our results are set out in Table 3.4 below:

**Table 3.4 Capital requirements**

<table>
<thead>
<tr>
<th>Unexpected Changes in Annual Mortality Improvement Rates</th>
<th>Capital Requirements ($m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>permanent assurances only : −0.5%</td>
<td>permanent assurances only : 65.8</td>
</tr>
<tr>
<td>immediate annuities only : +0.5%</td>
<td>immediate annuities only : 67.6</td>
</tr>
<tr>
<td>both policy types : +0.5% (or −0.5%)</td>
<td>both policy types : 3.0 (greater of the 2 cases)</td>
</tr>
<tr>
<td>permanent assurances only : −1.0%</td>
<td>permanent assurances only : 128.3</td>
</tr>
<tr>
<td>immediate annuities only : +1.0%</td>
<td>immediate annuities only : 140.8</td>
</tr>
<tr>
<td>both policy types : +1.0% (or −1.0%)</td>
<td>both policy types : 6.9 (greater of the 2 cases)</td>
</tr>
</tbody>
</table>
From this hypothetical example, it can be seen that the benefits of natural hedging can be substantial and an insurer can structure its business mix to reduce its capital requirements. While there is an implicit (simplified) assumption here that the unexpected changes in mortality improvement for the life insureds and annuitants are perfectly in line with each other, we have also tried other combinations (i.e. there is certain basis risk that the two groups have slightly different experiences) and the hedging effects are still considerable. We find that the correlation (across different ages) between the insured and annuitant improvement rates is 0.93 for males and is 0.54 for females (for ages 60 and over). After all, the two groups belong to the same population and mortality improvement generally happens to all socio-economic groups and ages, though to a different extent for different periods. It seems that in the longer run the experiences of the two groups would be more in line, while in the short run the relationship could be more volatile.

A stochastic approach which models both the insured and annuitant populations jointly could provide more insights into the assessment of natural hedging. One possibility is to utilise (13) in Section 2.2 to model the two populations coherently, and Li and Haberman (2012) are currently exploring the feasibility of this kind of approach.
3.7 Pricing methodologies

Pricing mortality or longevity linked securities generally involves constructing a mortality model, calibrating the model to existing market prices, and converting the real-world probability distribution into the risk-neutral probability distribution for pricing other securities. In an incomplete life market where very few securities are traded, there are infinitely many equivalent martingale measures. Two popular methods are using the Wang transform (e.g., Wang (2000)) and the Cairns, Blake, and Dowd (2006) model (15). The Wang transform uses the risk-neutral measure \( F^*(x) = \Phi(F^{-1}(x) - \lambda) \) and computes the market price of risk \( \lambda \) so that the discounted risk-neutral expected value of a future payment is equal to the current market price. As noted in Li (2010a), however, these pricing methods often need some subjective decisions, particularly when the model has two or more market prices of risk but there are not enough market prices available to compute them.

As an alternative, Kogure and Kurachi (2010) and Li (2010a) propose the maximum entropy approach, which has a number of advantages over the two pricing methods above. Firstly, it allows a variety of model structures and also the use of either bootstrapping or Bayesian MCMC simulation, which can incorporate process error, parameter error, and possibly model error. Secondly, it can readily be adapted to incorporate just one or more than one market price, without making the subjective decisions as stated above. Furthermore, it is theoretically sound and straightforward to extend this approach to cover more than one risk. In particular, Kogure, Li, and Kamiya (2012) propose a multivariate extension to price a reverse mortgage, which involves both longevity risk and house price deflation risk (see Section 5).

The key step of the maximum entropy approach is to minimise the Kullback-Leibler information criterion \( \sum \pi_j^* \ln(\pi_j^*/\pi_j) \), subject to two constraints \( \sum \text{PV}_j \pi_j^* = \text{MP} \) and \( \sum \pi_j^* = 1 \), in which \( \text{PV}_j \) is the present value of an insurance product or a security under scenario \( j \), MP is the corresponding current market price, \( \pi_j \) represents the real-world probabilities, and \( \pi_j^* \) represents the risk-neutral probabilities. The computed risk-neutral probabilities can then be used to price other securities. As mentioned, the random scenarios can be generated from one of the many model structures via MCMC simulation or bootstrapping, and more than one market price can be incorporated into the constraints.
3.8 The market conditions in Singapore

Recently, the Central Provident Fund (CPF) of Singapore has made it mandatory (effective 2013) for the citizens to annuitize a portion of their retirement assets in the national DC plan. As discussed in Section 3.5, mandatory annuitization can reduce adverse selection and increase market depth. The CPF has also entered the insurance market to provide these annuities, and this may reduce costs through economies of scale, enhance public confidence in buying annuities, and help avoid certain market distortions such as mis-selling. According to Fong, Mitchell, and Koh (2011), however, the money’s worth ratios (MWRs) were in the range of 0.86 to 1.08 for the life annuities (with fixed nominal payments) offered by the private insurers prior to the reform, while the MWRs are more than one for those offered by the CPF (assuming constant nominal payments). Moreover, seven out of the eight private insurers stopped offering CPF-compliant annuities between 2007 and 2009. Hence there are concerns about whether the annuities are too costly for the CPF to provide and whether there is any crowding out effect on the private insurers in the long term.

As noted in Fong, Mitchell, and Koh (2011), the life annuities offered by the CPF has a number of distinct features. First, there are four standard choices, called LIFE Basic, LIFE Balanced, LIFE Plus, and LIFE Income, which allow different levels of bequest. (These will reduce to only two choices in 2013.) Second, the future annuity payments may vary with the interest rates, investment performance, and mortality experience. Moreover, the amount that can be annuitized is capped at the stipulated minimum sum amount. Fong, Mitchell, and Koh (2011) suggest that the private sector can put their focus on wealthy individuals and their non-CPF assets. Apart from this direction, we also deem that the private sector can be more innovative when designing their products and provide more tailor-made, personal solutions, compared to the standard choices furnished by the CPF. As listed in Section 3.3, some overseas examples are inflation-linked annuities, variable annuities, participating annuities, impaired life annuities, and reverse mortgages, and many different features which aim to suit personal needs better can be incorporated. In particular, as discussed in Chia and Tsui (2005), housing assets composed 51% of the total assets in 2000 for Singaporeans, and a robust reverse mortgage market can help release the tied-up capital for financing retirement.

The reform of mandatory annuitization represents a significant move towards managing the continuing problem of longevity risk and population ageing. In particular, the
generous annuity pricing can act as a strong impetus for enhancing public awareness and initiating the development of the annuity market. In the longer run, the public would become more used to the notion of annuitization and gain a better understanding of it, and if there is any issue with financial sustainability, there may be a need to reduce the subsidisation gradually. Accordingly, more incentives can be provided to create a favourable environment for the longevity insurance market and the life market to develop, which would lead to more competition amongst insurers and better longevity solutions for both individuals and institutions. The new CPF scheme is at the forefront of the management of longevity and ageing issues in Asia-Pacific and its experience would serve as a very good reference for other policymakers in the region.
4  Mortality Indices (Working Paper 1)

In this section, we suggest that the time-varying parameters of the Cairns, Blake, and Dowd (2006) CBD model (15) can be used as indices to indicate levels of longevity risk over time. We study how these indices can be jointly modelled as multivariate time series which incorporate cross-correlations. We also construct the joint prediction region for the mortality indices. This region can provide a graphical risk metric for practitioners to compare the longevity risk exposures of different portfolios. This section is a working paper of Chan, Li, and Li (2012).

An introduction is given in Section 4.1. The CBD mortality indices are set forth in Section 4.2. The VARIMA modelling of the CBD mortality indices is covered in Section 4.3. The joint prediction region for the CBD mortality indices is illustrated in Section 4.4. Concluding remarks are given in Section 4.5.
4.1 Research background

Like stock indices, mortality indices can be built to indicate mortality levels over time. They have many potential applications. For example, their paths reflect the pace of mortality changes. The level of longevity risk can be measured from the differences between the expected and actual values of a mortality index. Moreover, one can construct standardised mortality-linked securities, such as longevity bonds and swaps, which can be traded between investors and pension plans who want to reduce their longevity risk exposures. Compared to customised instruments, standardised securities are easier for investors to understand and for the market to develop liquidity.

Several investment banks have established mortality indices in recent years, e.g. a longevity index by Credit Suisse in 2006, the QxX Index by Goldman Sachs and the LifeMetrics Index by JP Morgan in 2007, and the Xpect Cohort Index by Deutsche Börse in 2008. All these indices are non-parametric. Though it may be seen as an advantage, this kind of indices can carry only a limited amount of information. For example, an index based on life expectancy shows nothing about how the underlying mortality curve moves over time. Since these movements are often not uniform (e.g. Li and Luo (2012)), a pension plan or an annuity portfolio may find it difficult to construct a longevity hedge using such an index. To express the movements of a mortality curve in a non-parametric fashion, a large number of indices are needed, e.g. various indices linked to age-specific mortality rates. In this way, however, tracking of indices becomes tedious, and it is harder for the market to develop liquidity.

A model-based method can enhance the information content of mortality indices. In the past two decades, many extrapolative stochastic mortality models have been proposed in the literature. In general, when they are applied to past data, one or more series of time-varying parameters are estimated. By projecting these parameters into the future, a mortality forecast can be made. These parameters encompass much information about the changes of mortality across time and can be adopted as mortality indices. Our first objective here is to explore the feasibility of this parametric way of building mortality indices. This idea is somewhat parallel to the construction of the Chicago Board Options Exchange (CBOE) implied volatility indices, which are based on the volatility parameter in the Black-Scholes option pricing model.
Not all stochastic mortality models are suitable for creating indices. We deem that model-based mortality indices should meet three important criteria. Firstly, the indices should reflect the varying age-pattern of mortality improvement, not just the overall level. Secondly, the indices should have the new-data-invariant property, i.e. when the model is updated with an extra year of data, past index values are not affected. This feature allows one to track an index as its historical values remain unchanged. Thirdly, the indices should be clear and unambiguous to hedgers, investors, and even the general public. Among the six stochastic mortality models tested in Dowd et al. (2010a), we find that the original Cairns, Blake, and Dowd (2006) CBD model (15) seems to be the most suitable candidate. We argue that the two time-varying parameters $\kappa_{1,t}$ and $\kappa_{2,t}$ of the CBD model can be used jointly as mortality indices.

Our second objective is to co-model the CBD mortality indices over time. While Sweeting (2011) adopted a piecewise linear function, most researchers simply apply a bivariate random walk with drift to $\kappa_{1,t}$ and $\kappa_{2,t}$. The disadvantage is that it does not capture any serial- and cross-correlations. In the following, we demonstrate that these correlations between the two CBD mortality indices are significant. We then propose modelling the CBD mortality indices with vector autoregressive integrated moving average (VARIMA) models. We determine the VARIMA order through a multivariate version of the Box and Jenkins (1976) approach.

Our third objective is to find the joint prediction region for evaluating the variability of both CBD mortality indices. This prediction region can serve as a graphical risk metric and allow practitioners to compare the longevity risk exposures of different portfolios. It is a complement to the mortality fan chart (see Blake, Cairns, and Dowd (2008) and Dowd, Blake, and Cairns (2010)). The former describes the entire risk profile at a certain point of time, whereas the latter shows the variability of a particular death rate over time.
4.2 CBD mortality indices

The new-data-invariant property

The first objective is to create mortality indices from the time-varying parameters of a stochastic mortality model. We examine the six stochastic mortality models covered in Dowd et al. (2010a). We use the following notation:

- \( m_{x,t} \) is the central death rate at age \( x \) in year \( t \);
- \( q_{x,t} \) is the probability that a person aged \( x \) at time \( t \) will die between time \( t \) and \( t + 1 \);
- \( \beta^{(i)}_t \), \( i = 1, 2, 3 \), are age-specific parameters;
- \( \kappa^{(i)}_t \), \( i = 1, 2, 3 \), are time-varying parameters;
- \( \gamma^{(i)}_c \), \( i = 3, 4 \), where \( c = t - x \) is year of birth, are cohort-related parameters;
- \( n_a \) is the number of ages in the sample age range;
- \( \bar{x} \) is the mean age over the sample age range;
- \( \hat{\sigma}^2 \) is the mean of \( (x - \bar{x})^2 \) over the sample age range.

Table 4.1 lists the six mortality models. Most of them require setting constraints to ensure identifiability of parameters. The number of parameter constraints for each model is also shown in Table 4.1. Cairns et al. (2009) and Dowd et al. (2010a) discuss the identifiability problem in more detail.

We require the model to be entirely robust, i.e. when an extra year of data is available and the model is updated, past index values are not affected. This new-data-invariant property is very important because it would not be possible to track an index if its historical values are adjusted frequently. To examine this property, we fit the six models to English and Welsh male data collected from the Human Mortality Database (2012). We use the age range of 40-90, as some models cannot capture the accident hump at younger ages and the data beyond age 90 are sparse. We consider three sample periods: 1950-1989, 1950-1999, and 1950-2009. The three sets of parameter estimates during the overlapping years must be equal if the new-data-invariant property holds. The estimated parameters are plotted in Figure 4.1. It can be seen that the original CBD model (Model M5) is the only one that has the new-data-invariant property.
### Table 4.1 Six stochastic mortality models

| Model M1: The Lee-Carter model | $\ln m_{x,t} = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)}$ | (2 constraints) |
| Model M2: The Renshaw-Haberman model | $\ln m_{x,t} = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)} + \beta_x^{(3)} \gamma_t^{(3)}$ | (4 constraints) |
| Model M3: The age-period-cohort model | $\ln m_{x,t} = \beta_x^{(1)} + n_a^{-1} k_t^{(2)} + n_a^{-1} \gamma_t^{(3)}$ | (3 constraints) |
| Model M5: The original CBD model | $\ln \frac{q_x}{1-q_x} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x})$ | (no constraint) |
| Model M6: The CBD Model with a cohort effect term | $\ln \frac{q_x}{1-q_x} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_t^{(3)}$ | (2 constraints) |
| Model M7: The CBD Model with cohort effect and quadratic terms | $\ln \frac{q_x}{1-q_x} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \kappa_t^{(3)}((x - \bar{x})^2 - \sigma_x^2) + \gamma_t^{(4)}$ | (3 constraints) |

We note that the original CBD model may not lead to the best goodness-of-fit among the six choices. But our main concern is the tractability of the mortality indices. We prefer a model that is completely robust rather than all-rounded. The use of the CBD model to produce mortality indices may be compared with the use of the Black-Scholes model to produce implied volatility indices. We know that both models may not be completely accurate, but they do produce indices that are tractable by investors and other users.
Figure 4.1  Estimates of the time-varying parameters of the six stochastic mortality models for 1950-1989 (solid), 1950-1999 (dotted), and 1950-2009 (dashed)

Intuitions behind the CBD mortality indices

The CBD model specifies that for a given year $t$, the logit of $q_{x,t}$ is linearly related to age $x$. The first CBD mortality index $\kappa^{(1)}_t$ refers to the level of the logit mortality curve in year $t$. A reduction in $\kappa^{(1)}_t$, i.e. a parallel downward shift of the curve, represents an overall mortality improvement. The impact of a reduction in $\kappa^{(1)}_t$ is shown in the left panel of Figure 4.2. The second CBD mortality index $\kappa^{(2)}_t$ refers to the slope of the logit mortality curve in year $t$. An
increase in $\kappa_i(2)$, i.e. an increase in the steepness of the curve, implies that mortality at younger ages (below the mean age $\bar{x}$) improves faster than at older ages (above the mean age $\bar{x}$). The impact of an increase in $\kappa_i(2)$ is shown in the middle panel of Figure 4.2. Note that when the first index is fixed, a change in the second index has no impact at the mean age $\bar{x} = 65$. The right panel of Figure 4.2 demonstrates the impact when both indices are varied. In Figure 4.3 we plot the mortality curves in the original scale for four different pairs of the indices. It can be seen that the two indices can jointly capture different patterns of the underlying mortality curve.

It is important to consider the two indices jointly because the association between them has a significant impact on the longevity risk exposure of a portfolio. We now consider two examples. Firstly, we consider a closed pension plan. Its payouts are larger when the overall mortality improvement is higher than expected, i.e. future $\kappa_i(1)$ are lower than expected. Moreover, for a fixed overall mortality improvement, the problem would be worse when mortality at older ages improves faster, i.e. future $\kappa_i(2)$ are lower than expected. Secondly, we consider an insurance company which sells term-life insurances to young people. Its payouts are greater when the overall mortality improvement is smaller than expected, i.e. future $\kappa_i(1)$ are greater than expected. Also, for a fixed overall mortality improvement, the problem would be worse when mortality at older ages improves faster, i.e. future $\kappa_i(2)$ are smaller than expected.

We have illustrated that the CBD mortality indices meet the three important criteria:

I. The indices can reflect the varying age-pattern of mortality improvement, not just the overall level.

II. The indices have the new-data-invariant property.

III. The indices have clear interpretations.
Figure 4.2  Movements of logit mortality curve when either one or both of the CBD mortality indices vary

Figure 4.3  Mortality curves in original scale for different pairs of CBD mortality indices
4.3 Vector time series modelling

In earlier studies of the CBD model, the time-varying parameters are often assumed to follow a bivariate random walk with drift. This simple process, however, does not fully capture the association between the two mortality indices. This problem can be tackled by using a vector autoregressive integrated moving average (VARIMA) process instead. The VARIMA order can be found by a multivariate version of the Box and Jenkins (1976) approach. More details are given in Tiao and Box (1981) and Wei (2006). In the following, we continue to use English and Welsh male data of the age range 40-90 and the sample period 1950-2009.

Model specification

The vector autoregressive moving average VARMA\((p, q)\) process for a bivariate vector time series \(Z_t = (Z_t^{(1)}, Z_t^{(2)})'\) is specified as:

\[
Z_t = C_0 + \sum_{i=1}^{p} \Phi_i Z_{t-i} + \sum_{j=1}^{q} \Theta_j \epsilon_{t-j} + \epsilon_t,
\]

in which \(C_0\) is a \((2 \times 1)\) intercept vector, and \(\Phi_i\) and \(\Theta_j\) are \((2 \times 2)\) autoregressive and moving average coefficient matrices respectively. The residual \((2 \times 1)\) vectors \(\epsilon_t\) are independently and identically distributed as bivariate normal with mean zero and variance-covariance matrix \(\Sigma\). The duplet \((p, q)\) is the order of the VARMA process.

The cross-correlation matrices are denoted as \(\Gamma(l)\) for \(l = 0, 1, \ldots\) The \((i, j)th\) element of \(\Gamma(l)\) is the correlation coefficient between \(Z_t^{(i)}\) and \(Z_{t-l}^{(j)}\). When \(p = 0\), i.e. \(Z_t\) is a vector MA\((q)\) process, the cross-correlation matrices for \(l > q\) are all zero. On the other hand, the partial autoregression matrices are denoted as \(P(l)\) for \(l = 0, 1, \ldots\) The partial autoregression matrix \(P(l)\) at lag \(l\) is equal to \(\Phi_{t,l}\) of the vector AR\((l)\) process below:

\[
Z_{t+l} = \Phi_{l,1} Z_{t+1} + \Phi_{l,2} Z_{t+2} + \ldots + \Phi_{l,l} Z_t + \epsilon_{t+l},
\]

When \(q = 0\), i.e. \(Z_t\) is a vector AR\((p)\) process, the partial autoregression matrices for \(l > p\) are all zero.

The VARMA process above is suitable only if \(Z_t\) is weakly stationary, i.e. both the mean vector and covariance matrix of \(Z_t\) are constant over time. Otherwise, we may model \(\Delta Z_t = Z_t - Z_{t-1}\) instead. If \(\Delta Z_t\) is still non-stationary, we try \(\Delta^2 Z_t = \Delta Z_t - \Delta Z_{t-1}\), and so on. Let \(d\) be the number of differencing required to achieve stationarity. The resulting process,
\[
\Delta^d Z_t = C_0 + \sum_{i=1}^{p} \Phi_i \Delta^d Z_{t-i} + \sum_{j=1}^{d} \Theta_j \epsilon_{t-j} + \epsilon_t ,
\]

is a vector autoregressive integrated moving average VARIMA\((p, d, q)\) process. The triplet \((p, d, q)\) is the order of the VARIMA process.

**Model identification**

The multivariate version of the Box and Jenkins approach involves the ‘cut-off’ properties of \(\Gamma(l)\) and \(P(l)\), which are estimated as the sample cross-correlation matrix (SCCM) and the sample partial autoregression matrix (SPAM) respectively. The \((i, j)\)th element of the lag-\(l\) SCCM \(\hat{\Gamma}(l)\) can be computed as the sample correlation coefficient between \(Z_{d,i}^{(l)}\) and \(Z_{d,j}^{(l)}\). If the VARMA process holds, the standard error of each element of the SCCM is approximately \(1/\sqrt{n}\). We can use this property to test the significance of \(\Gamma(l)\) and find \(q\) in the VARIMA order. Note that if \(Z_t\) is non-stationary, the SCCM will show no decay to zero. Moreover, the lag-\(l\) SPAM \(\hat{P}(l)\) and the standard errors of its elements can be computed by fitting the vector AR\((l)\) process above. Tiao and Box (1981) suggest using the likelihood ratio statistic to test the null hypothesis \(P(l) = 0\) versus the alternative \(P(l) \neq 0\). The term \(\Xi(l) = \left| \hat{\Sigma}(l) \right| / \left| \hat{\Sigma}(l-1) \right|\) is first calculated, in which \(\hat{\Sigma}(l)\) is the matrix of residual sum of squares and cross products after fitting the vector AR\((l)\). Using Bartlett’s (1938) approximation, the likelihood statistic \(M(l) = -\left( n - 5l - \frac{5}{2} \right) \ln \Xi(l) \) is asymptotically distributed as \(\chi^2\) with four degrees of freedom under the null hypothesis. The SPAM helps us find \(p\) in the VARIMA order.

Tiao and Box (1981) summarise these matrix elements with symbols +, −, and •, in which + (−) represents a value greater (less) than twice the estimated standard error and • represents an insignificant value. We now set \(Z_t\) as \((\kappa_t^{(1)}, \kappa_t^{(2)})\)' and calculate its SCCM and SPAM, as shown in Table 4.2. As the SCCM shows no decay to zero, \(Z_t\) is non-stationary. We then replace \(Z_t\) with \(\Delta Z_t\) and calculate its SCCM and SPAM, as shown in Table 4.3. The SCCM now cuts off at lag 1 and so no further differencing is needed. The \(M\)-statistic has a critical value of \(\chi^2_{4,0.95} = 9.45\) at 5% significance level and the SPAM cuts off at lag 5. These results suggest either VARIMA\((0,1,1)\) or VARIMA\((5,1,0)\). As the fitted VARIMA\((0,1,1)\) does not pass the diagnostic checks on the residuals, we choose VARIMA\((5,1,0)\).
Table 4.2  The SCCM and SPAM for the CBD mortality indices

<table>
<thead>
<tr>
<th>lag (l)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
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<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>-</td>
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</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
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<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Sample cross-correlation matrices (SCCM)

\[
\begin{pmatrix}
- & + \\
+ & - \\
+ & - \\
+ & - \\
+ & - \\
+ & - \\
+ & - \\
+ & - \\
\end{pmatrix}
\]

(b) Sample partial autoregression matrices (SPAM)

\[
\begin{pmatrix}
+ & - \\
+ & + \\
+ & + \\
+ & + \\
+ & + \\
+ & + \\
+ & + \\
+ & + \\
\end{pmatrix}
\]

\[
M(l) \quad 319.70 \quad 15.47 \quad 9.87 \quad 5.46 \quad 6.24 \quad 9.13 \quad 6.68 \quad 1.58
\]

Table 4.3  The SCCM and SPAM for the CBD mortality indices after first order differencing

<table>
<thead>
<tr>
<th>lag (l)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Sample cross-correlation matrices (SCCM)

\[
\begin{pmatrix}
- & - \\
- & - \\
- & - \\
- & - \\
- & - \\
- & - \\
- & - \\
- & - \\
\end{pmatrix}
\]

(b) Sample partial autoregression matrices (SPAM)

\[
\begin{pmatrix}
- & - \\
- & - \\
+ & + \\
+ & + \\
+ & + \\
+ & + \\
+ & + \\
+ & + \\
\end{pmatrix}
\]

\[
M(l) \quad 12.42 \quad 1.88 \quad 8.08 \quad 7.31 \quad 14.88 \quad 7.79 \quad 4.98 \quad 7.81
\]
Model estimation

The VARIMA parameters are estimated via conditional maximum likelihood (see Wilson (1973)). The parameter estimates and standard errors are given in Part (a) of Table 4.4. We set the insignificant parameters as zero and re-estimate the remaining parameters via the exact likelihood method (see Reinsel (1997), Chapter 5). The final parameter estimates are given in Part (b) of Table 4.4.

Table 4.4  Fitted VARIMA (5,1,0) for the CBD mortality indices

(a) Full Model

<table>
<thead>
<tr>
<th></th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0$</td>
<td>$-0.09$</td>
<td>$-0.26$</td>
<td>$-0.26$</td>
</tr>
<tr>
<td></td>
<td>$(0.006)$</td>
<td>$(1.49)$</td>
<td>$(3.05)$</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>$\Phi_4$</td>
<td>$\Phi_5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.01$</td>
<td>$0.01$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(0.006)$</td>
<td>$(0.14)$</td>
<td></td>
</tr>
</tbody>
</table>

(b) Final Model

<table>
<thead>
<tr>
<th></th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0$</td>
<td>$-0.08$</td>
<td>$-0.37$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$(0.04)$</td>
<td>$(1.23)$</td>
<td>$(1.00)$</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>$\Phi_4$</td>
<td>$\Phi_5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.01$</td>
<td>$0.01$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(0.006)$</td>
<td>$(0.12)$</td>
<td></td>
</tr>
</tbody>
</table>

Fitted VARIMA (5,1,0) for the CBD mortality indices
4.4 A graphical risk metric

K1 and K2 risks

Based on the CBD model, we divide longevity risk into two components: K1 risk and K2 risk. The former arises from the uncertainty in \( \kappa_t^{(1)} \) and is associated with overall mortality improvements. The latter arises from the uncertainty in \( \kappa_t^{(2)} \) and is associated with the differentials across different age groups. In Figure 4.4, the horizontal and vertical axes represent the values of \( \kappa_t^{(1)} \) and \( \kappa_t^{(2)} \) in a future year \( T \). The K1 and K2 risks can be seen as the random variations along the horizontal and vertical dimensions from the best estimates \( \hat{\kappa}_T^{(1)} \) and \( \hat{\kappa}_T^{(2)} \). For example, the lower left quadrant is the worst outcome for pension plans, as discussed in Section 4.2. On the other hand, the lower right quadrant is the worst outcome for life insurers selling term life insurances to younger people.

Figure 4.4 K1 and K2 risks

The concept of joint prediction regions

We call \( J_t \) a joint prediction region for the time-\( t \) values of the CBD mortality indices with coverage probability \( 0 < 1 - \alpha \leq 1 \) if \( \Pr((\kappa_t^{(1)}, \kappa_t^{(2)}) \in J_t) = 1 - \alpha \). The region \( J_t \) contains...
100(1 – α)% of the possible combinations of the two indices at time \( t \). This region can serve as a graphical metric of longevity risk. Firstly, the area refers to the aggregate level of longevity risk at time \( t \). Secondly, the shape refers to the risk profile of a particular financial institution. Two examples are plotted in Figure 4.5. The areas of these two regions are similar, which means similar extent of aggregate longevity risk. However, Example 1 has more risk for a closed pension plan, whereas Example 2 has more risk for an insurance company selling term life insurances to younger people.

**Figure 4.5** Two hypothetical joint prediction regions for the CBD mortality indices

![Diagram](image)

**Constructing joint prediction regions**

For a fixed \( \alpha \), the joint prediction region is not unique. There are different methods for constructing joint prediction regions (e.g. Lutkepohl (1991), Chan, Cheung, and Wu (1999)). We propose to use a numerical method as follows:

- Calculate \( \hat{\kappa}_{t}^{(1)} \) and \( \hat{\kappa}_{t}^{(2)} \) by switching off the random components of the VARIMA model.
- From the VARIMA model, simulate \( N \) pairs of \( \kappa_{t}^{(1)} \) and \( \kappa_{t}^{(2)} \).
- Calculate \( s_{t}^{(1)} \) and \( s_{t}^{(2)} \), the standard deviations of the simulated \( \kappa_{t}^{(1)} \)'s and \( \kappa_{t}^{(2)} \)'s.
- For each simulated pair of \( \kappa_{t}^{(1)} \) and \( \kappa_{t}^{(2)} \), calculate its weighted distance to the best estimate as:
\[
\sqrt{\left(\frac{\kappa^{(1)}_i - \hat{\kappa}^{(1)}_i}{s^{(1)}_i}\right)^2 + \left(\frac{\kappa^{(2)}_i - \hat{\kappa}^{(2)}_i}{s^{(2)}_i}\right)^2}.
\]

- Sort the \( N \) simulated pairs of \( \kappa^{(1)}_i \) and \( \kappa^{(2)}_i \) by their distances to the best estimate. Choose the \( (1 - \alpha)N \) pairs with the shortest distances.
- Draw a convex hull to enclose the \( (1 - \alpha)N \) chosen pairs of simulated \( \kappa^{(1)}_i \) and \( \kappa^{(2)}_i \). Geometrically speaking, the convex hull is the smallest convex set that contains the selected \( (1 - \alpha)N \) points. The use of a convex hull prevents the joint prediction region from overstating the underlying uncertainty.
- The convex hull drawn is a \( 100(1 - \alpha)\% \) joint prediction region for the two CBD mortality indices. It contains a randomly selected pair of indices in the simulated sample with probability (no less than) \( 1 - \alpha \). Note that the coverage probability might be slightly greater than \( 1 - \alpha \) due to rounding.
4.5 Concluding remarks

In this section, we suggested using $\kappa^{(1)}_t$ and $\kappa^{(2)}_t$ of the original CBD model as mortality indices, because they possess three important properties: (1) they can reflect a varying age-pattern of mortality improvement; (2) they are new-data-invariant; (3) they are clear and unambiguous. We then modelled the two indices jointly with a VARIMA process, which can capture the serial- and cross-correlations between the two indices. We also described the variability of the two indices by the joint prediction region. This region can be used as a graphical risk metric to allow practitioners to compare the longevity risk exposures of different portfolios.

One possible future research is to introduce structural breakpoints (Li, Chan, and Cheung (2011)) and outliers (Li and Chan (2005, 2007)). Another possibility is to include parameter uncertainty via Bayesian approaches, in which parameters are treated as random variables.

Like q-forward and S-forward (Coughlan (2009)), we propose a standardised security called K-forward, which is a zero-coupon swap that exchanges on the maturity date a fixed amount for a random amount linked to a CBD mortality index at some future time. For the fixed receiver, the payoff from a K-forward is $X \times (\bar{\kappa}^{(i)}_{t*} - \kappa^{(i)}_t)$, $i = 1, 2$, in which $X$ is the notional amount, $t*$ is the maturity date, and $\bar{\kappa}^{(i)}_{t*}$ is the forward value. A pension plan may write K-forwards as a fixed receiver. Payouts from the K-forwards can offset the worst outcome, which occurs when both indices turn out to be lower than expected. The calibration may be done with adapted versions of the existing mortality duration and convexity measures. For example, Cairns (2011) derived semi-analytic formulae for calculating the ‘deltas’ of an annuity portfolio with respect to the two time-varying parameters of the CBD model.
5  Multivariate Risk-Neutral Pricing Method (Working Paper 2)

In this section, we propose a Bayesian multivariate framework for pricing reverse mortgages which contain a number of insurance and financial risks, such as mortality improvement and house price growth. Our approach is a multivariate extension of Kogure and Kurachi (2010). We apply it to Japanese data and the results suggest some promising future for the market. This section is a working paper of Kogure, Li, and Kamiya (2012).

An introduction is given in Section 5.1. A method to translate the bivariate predictive distribution into its risk-neutral form is set forth in Section 5.2. A brief discussion of reverse mortgage plans is provided in Section 5.3. A Bayesian Lee-Carter model is fit in Section 5.4 and a local level model is fit in Section 5.5 to Japanese data. The numerical results are presented in Section 5.6. Concluding remarks are given in Section 5.7.
5.1 Research background

A reverse mortgage has been regarded as a useful tool for managing a person’s longevity risk. It involves several risks, such as longevity, house prices, and interest rates. To price reverse mortgage plans, a multivariate framework is required. In this section, we propose a Bayesian multivariate method by generalizing the univariate method of Kogure and Kurachi (2010) and apply it to Japanese data. We assume that interest rates are constant and deduce the bivariate predictive distribution of house prices and mortality rates.

There have been a number of pricing methods proposed in the literature, e.g. Wang, Valdez, and Piggott (2008) and Chen, Cox, and Wang (2010). Our method is different to these as we adopt a fully Bayesian approach. We construct the predictive distributions of the risk factors and then convert them into risk-neutral form via the maximum entropy principle. By adopting this Bayesian approach we can take care of parameter uncertainty in the pricing. Readers may refer to Cairns, Blake, and Dowd (2006) for discussion on parameter uncertainty in modelling longevity risk.
5.2 Bayesian risk-neutral method

Risk-neutral distributions

We first consider a period from the current time 0 to a time point \( T > 0 \) in the future. Let \( X \) be the value of a risk factor at time \( T \) and \( F_X \) be the distribution function of \( X \). There is a derivative which pays off \( C(X) \) at time \( T \). We assume that the discount rate \( r \) is constant over the period. The present value of the derivative at time 0 is:

\[
V = e^{-rT} \int C(x) \, dF_X^* ,
\]

where \( F_X^* \) is the risk-neutral distribution function of \( X \) which is deduced as:

\[
dF_X^*(x) = p(x) \, dF_X(x) ,
\]

with a state price density \( p(x) \).

Under the standard option pricing theory, the concept of completeness is developed under the no-arbitrary principle to find the state price density \( p \) and thus the risk-neutral distribution \( F_X^* \). But it would be inappropriate to assume such completeness for non-financial risks such as mortality risk. In the insurance risk theory, the state price density is selected according to other conditions. One popular choice is the Esscher transform in which the state price density is:

\[
p(x) = \frac{e^{\gamma x}}{E_X(e^{\gamma X})} .
\]

Note that \( E_X \) is the expectation under \( F_X \) and \( \gamma \) is a constant based on some market conditions. The other one is the Wang transform in which the state price density is:

\[
p(x) = \frac{\phi(\xi - \gamma)}{\phi(\xi)} , \quad \xi = \Phi^{-1}(F_X(x)) ,
\]

where \( \Phi \) is the standard normal distribution function and \( \phi \) is its density. See Gerber and Shiu (1994) for the Esscher transform and Wang (2000) for the Wang transform.

Bayesian risk-neutral distribution

In practice, the distribution function \( F_X \) is unknown and we assume a certain statistical model \( \{ f(\cdot|\theta), \theta \in \Theta \} \) with \( f_X(x) = dF_X(x)/dx \). Then the formula of the present value above
becomes:

$$V(\theta) = e^{-rT} \int C(x) f^*(x|\theta) dx ,$$

with $f^*(x|\theta) = p(x) f(x|\theta)$.

In the standard frequentist approach, the maximum likelihood estimator $\hat{\theta}$ is obtained from data $D_0$ and incorporated into above to yield:

$$V(\hat{\theta}) = e^{-rT} \int C(x) f^*(x|\hat{\theta}) dx .$$

When the specification $\{ f(\cdot|\theta), \theta \in \Theta \}$ has many parameters, parameter uncertainty cannot be ignored. Then it would be better to use the Bayesian approach. With a posterior density $g(\theta|D_0)$, the Bayesian pricing formula is:

$$\bar{V} = \int V(\theta) g(\theta|D_0) d\theta$$

$$= e^{-rT} \int C(x) \int f^*(x|\theta) g(\theta|D_0) d\theta dx$$

$$= e^{-rT} \int C(x) f^*(x|D_0) dx .$$

Here, $f^*(x|D_0) = \int f^*(x|\theta) g(\theta|D_0) d\theta = p(x) f(x|D_0)$, $f(x|D_0) = \int f(x|\theta) g(\theta|D_0) d\theta$.

So $f^*(x|D_0)$ is the risk-neutral version of the Bayesian predictive density $f(x|D_0)$.

Risk-neutralization of predictive distributions

Suppose there are $m$ securities. For each $1 < i < m$, let $h_i(x)$ and $v_i$ be the payoff function and the present value of the $i$th security. Then under the risk-neutral density $f^*(x|D_0)$, there are moment conditions:

$$E[h_i(X)|D_0] = \int h_i(x) f^*(x|D_0) dx = a_i ,$$

with $a_i = e^{rT}v_i$ for $i = 1, 2, \ldots, m$. Stutzer (1996) suggested to use $f^*(x|D_0)$ that minimises the Kullback-Leibler information:

$$\int f^*(x|D_0) \ln \frac{f^*(x|D_0)}{f(x|D_0)} dx ,$$

subject to the moment conditions above with the constraint:

$$\int f^*(x|D_0) dx = 1 .$$

The resulting density is:
\[ f^*(x \mid D_0) = f(x \mid D_0) \exp \left\{ \gamma_0 + \gamma_1 h_1(x) + ... + \gamma_m h_m(x) \right\}, \]
in which \( \gamma_0, \gamma_1, \ldots, \gamma_m \) are computed by the moment conditions and the constraint.

In addition to \( X \), suppose we have another risk factor \( Y \). Let \( f(x, y \mid D_0) \) be the two-dimensional predictive density. There are \( m \) securities and for each \( 1 < i < m \), let \( h_i(x,y) \) and \( v_i \) be the payoff function and the present value of the \( i \)th security. Following the idea of Stutzer (1996), we minimise the two-dimensional cross-entropy:

\[
\int \int f^*(x, y \mid D_0) \ln \frac{f^*(x, y \mid D_0)}{f(x, y \mid D_0)} \, dx \, dy,
\]
with respect to \( f^*(x, y \mid D_0) \) subject to the moment conditions:

\[
\mathbb{E}^*[h_i(X,Y)\mid D_0] = \int \int h_i(x, y) f^*(x, y \mid D_0) \, dx \, dy = a_i,
\]
with \( a_i = e^{\mathbf{r}^T \mathbf{v}_i} \) for \( i = 1, 2, \ldots, m \) and the constraint:

\[
\int \int f^*(x, y \mid D_0) \, dx \, dy = 1.
\]

The resulting density is:

\[
f^*(x, y \mid D_0) = f(x, y \mid D_0) \exp \left\{ \gamma_0 + \gamma_1 h_1(x, y) + ... + \gamma_m h_m(x, y) \right\},
\]
in which \( \gamma_0, \gamma_1, \ldots, \gamma_m \) are computed by the moment conditions and the constraint. It should be noted that \( X \) and \( Y \) may be dependent under this risk-neutral distribution even if they are independent under \( f(x, y \mid D_0) \). However, if each \( h_i \) depends on either \( x \) or \( y \), but not both, then \( X \) and \( Y \) are independent under this risk-neutral distribution if they are so under \( f(x, y \mid D_0) \). More generally, this will be true if each \( h_i \) is additive.
5.3 Reverse mortgages

The non-recourse provision

A reverse mortgage loan is non-recourse, i.e. the property owner’s obligation is limited to the proceeds from the sale of the property. Consider a person aged \( x \) who enters into a reverse mortgage contract at time 0. Let \( L_t \) be the outstanding balance of the loan and \( H_t \) be the value of the mortgaged property at time \( t \). If the borrower dies in year \( t \), the loan is terminated and the lender will receive \( \min(L_t, H_t) = L_t - \max(L_t - H_t, 0) \). Suppose \( L_t = L_0e^{ut} \) with \( u \) being the interest rate charged on the mortgage loan. The term \( \max(L_t - H_t, 0) \) represents a cash flow from the non-recourse provision. It can be seen as a short put option on the house price \( H_t \) with a strike price \( L_t \).

Evaluation of reverse mortgages

Consider a cohort group of age \( x \). Let \( \omega \) be the highest attained age and set \( T = \omega - x \). Moreover, let \( I_t \) be the proportion of the cohort who will die in year \( t \). The per capita cash flow from the reverse mortgages is given as \( (L_t - \max(L_t - H_t, 0)) \times I_t \), \( t = 1, 2, \ldots, T \). We assume that \( I_t \) and \( H_t \) are random and discount rates are constant. Let \( d(t) = e^{-rt} \) be the \( t \)-year discount factor. Then the average value of the reverse mortgages is:

\[
E^* \left[ \sum_{t=0}^{T} d(t)(L_t - \max(L_t - H_t, 0))I_t \right] = \sum_{t=0}^{T} d(t)L_t E^*[I_t] - \sum_{t=0}^{T} d(t)E^*\left[ \max(L_t - H_t, 0)I_t \right].
\]

Here \( E^* \) denotes the expectation under the risk-neutral measure. This value must be greater than \( L_0 \) for the reverse mortgages to be sustainable.

We assume that \( I_t \) and \( H_t \) are independent under the original (physical) predictive distribution. Then, in light of the discussion made earlier, they also become independent under the risk-neutral predictive distribution provided that all the moment constraints are additive. Then the second term on the right hand side can be formulated as:

\[
\sum_{t=0}^{T} d(t)E^*[I_t]E^*\left[ \max(L_t - H_t, 0) \right],
\]

and the valuation requires separate calculation of \( E^*[I_t] \) and \( E^*\left[ \max(L_t - H_t, 0) \right] \).
5.4 Bayesian pricing of $l_t$

We set $l_t = q_{tx} q_{tx+1}(t) = q_{tx} - q_{tx+1} p_x$, where $q_{tx}$ is the probability that a person aged $x$ at time 0 will be alive at time $t$ and $q_{tx+1}(t)$ is the probability that a person aged $x$ at time $t$ will die within one year. Since $q_{tx}$ is written as:

$$q_{tx} = (1 - q_x(0)) \times (1 - q_{x+1}(1)) \times \ldots \times (1 - q_{x+t-1}(t-1)),$$

we consider modelling $q_{tx}(t)$.

The Lee-Carter model

Following Lee and Carter (1992), we assume that the force of mortality $\mu_x(t)$ is modelled as $\mu_x(t) = \exp(\alpha_x + \beta_x \kappa_t)$. We further assume that $\mu_{x+t}(t + u) = \mu_x(t)$ for integers $x$ and $t$, and for all $0 \leq s, u < 1$. The death probability is $q_x(t) = 1 - \exp\{- \exp(\alpha_x + \beta_x \kappa_t)\}$.

Modelling of mortality

Let $m_{xt}$ be the crude death rate $m_{xt} = D_{xt} / E_{xt}$, where $D_{xt}$ is the number of deaths at age $x$ in year $t$, and $E_{xt}$ is the number of person years from which $D_{xt}$ occurred. The Lee-Carter method assumes that $y_{xt} = \ln m_{xt}$ follows a regression model $y_{xt} = \alpha_x + \beta_x \kappa_t + \varepsilon_{xt}$ for $x = x_{\min}, x_{\min} + 1, \ldots, x_{\max}$ and for $t = t_{\min}, t_{\min} + 1, \ldots, t_{\max}$, in which the error terms $\varepsilon_{xt}$’s are distributed independently and identically as normal with zero mean and constant variance. The parameters $\{\beta_x\}$ and $\{\kappa_t\}$ are subject to $\sum_{x=x_{\min}}^{x_{\max}} \beta_x = 1$ and $\sum_{t=t_{\min}}^{t_{\max}} \kappa_t = 0$.

Let $y_t = (y_{x_{\min}}, \ldots, y_{x_{\max}})'$ and set up a state-space model with observation equation $y_t = \alpha + \beta \kappa_t + \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2 I_M)$, and state equation $\kappa_t = \lambda + \kappa_{t-1} + \omega_t$, $\omega_t \sim N(0, \sigma_\omega^2)$, where $\alpha = (\alpha_{x_{\min}}, \ldots, \alpha_{x_{\max}})'$, $\beta = (\beta_{x_{\min}}, \ldots, \beta_{x_{\max}})'$, $M = x_{\max} - x_{\min} + 1$, and $I_M$ is the identity matrix of dimension $M \times M$. Then the likelihood function is:

$$l(y | \alpha, \beta, \kappa, \lambda, \sigma_\varepsilon^2, \sigma_\omega^2) = \prod_{t=t_{\min}}^{t_{\max}} \prod_{x=x_{\min}}^{x_{\max}} f(y_{xt} | \alpha, \beta_x, \kappa_t, \sigma_\varepsilon^2)$$

$$\propto \left( \frac{1}{\sigma_\varepsilon} \right)^{LM} \exp \left\{ -\frac{1}{2\sigma_\varepsilon^2} \sum_{t=t_{\min}}^{t_{\max}} \sum_{x=x_{\min}}^{x_{\max}} (y_{xt} - (\alpha_x + \beta_x \kappa_t))^2 \right\},$$

where $L = t_{\max} - t_{\min} + 1$. 

2012 Insurance Risk and Finance Research Centre 74
The prior distributions for $\alpha$ and $\beta$ for the observation equation are chosen as $\alpha \sim N(0_M, \sigma_\alpha^2 I_M)$, $\beta \sim N((1/M)I_M, \sigma_\beta^2 I_M)$, in which $0_M$ and $I_M$ are the $M$-dimensional vectors of all zeros and all ones respectively. The underlined characters denote hyperparameters. The prior for $\sigma_\alpha^2$ is set up as $\sigma_\alpha^2 \sim IG(a_\alpha, b_\alpha)$, where $IG(a, b)$ is the inverse Gamma distribution with shape parameter $a$ and scale parameter $b$. The priors for $\lambda$ and $\sigma_\omega^2$ in the state equation are chosen as $\lambda \sim N(\lambda_0, \sigma_\lambda^2)$ and $\sigma_\omega^2 \sim IG(\sigma_\omega, b_\omega)$. Also, the prior for $\kappa_{\text{min}}$, the first element of $\kappa_t$’s, is chosen as $\kappa_{\text{min}} \sim N(c, R)$.

Risk-neutral predictive distribution

We generate $N$ paths of outcomes from MCMC sampling and denote them as $(p_{x}^{(j)} = (p_{x}^{(j)}, 2p_{x}^{(j)}, \ldots, np_{x}^{(j)}), j = 1, 2, \ldots, N)$. Consider a life annuity in which a person receives a constant payment of $1$ at the end of each year over $T$ years as long as he is alive. Then the present value of the annuity for a person aged $x$ at time $0$ is $a_x = \sum_{t=1}^{T} d(t) \cdot p_x$. Let $a_x^{\text{market}}$ be the market value of the annuity. Then for a probability $\pi^*$ over $\{p_{x}^{(j)}\}$ to be risk-neutral, it must satisfy $\sum_{j=1}^{N} a_x^{(j)} \pi_ j^* = a_x^{\text{market}}$, with $a_x^{(j)} = \sum_{t=1}^{T} d(t) \cdot p_x^{(j)}$. Let $\pi$ be the empirical distribution of the $N$ paths of the MCMC sampling, which has an equal probability of $1/N$ for each path. Then the maximum entropy principle stipulates that the risk-neutral distribution $\pi^*$ should minimise the Kullback-Leibler information divergence $\sum_{j=1}^{N} \pi_ j^* \ln(\pi_ j^*/\pi_ j)$, subject to the market value condition above and $\sum_{j=1}^{N} \pi_ j^* = 1$, $\pi_ j^* > 0$ for $j = 1, 2, \ldots, N$. The solution to this constrained minimisation is:

$$\hat{\pi}_ j^* = \frac{\pi_ j \exp(\gamma a_x^{(j)})}{\sum_{j=1}^{N} \pi_ j \exp(\gamma a_x^{(j)})} \quad \text{for } j = 1, 2, \ldots, N,$$

in which $\gamma$ is computed from:

$$a_x^{\text{market}} = \frac{\sum_{j=1}^{N} a_x^{(j)} \exp(\gamma a_x^{(j)})}{\sum_{j=1}^{N} \exp(\gamma a_x^{(j)})}.$$

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Evaluation of $I_t$ for Japanese data

The data we used are the number of deaths and populations of Japan between ages 0 and 98 over the period from 1973 to 2008 for both sexes, obtained from the Population Census and the Vital Statistics of Japan. We performed 25,000 steps of MCMC simulation. We discarded the first 5,000 steps and used the remaining 20,000 steps to eliminate initial effects. We adopted the following setup: age of cohort $x = 65, 70, 75, 80, 85, \text{ and } 90$, and discount factor $d(t) = e^{-rt}$, $r = 0.015$. To determine the market price, $a_t^{\text{market}}$, we used the death probabilities for annuity products tabulated in the ‘standard life table 2007’ constructed by the Institute of Actuaries of Japan. Most of Japanese life insurance companies price their insurance products using the standard life table 2007. Figures 5.1(a) to 5.1(f) plot $E^*[I_t]$ and $E[I_t]$. 
Figure 5.1(a) The changes of $E^*[I_t]$ (solid line) and $E[I_t]$ (dashed line) with $t$ for male (left panel) and female (right panel) populations of age 65

Figure 5.1(b) The changes of $E^*[I_t]$ (solid line) and $E[I_t]$ (dashed line) with $t$ for male (left panel) and female (right panel) populations of age 70
Figure 5.1(c) The changes of $E^*[I_t]$ (solid line) and $E[I_t]$ (dashed line) with $t$ for male (left panel) and female (right panel) populations of age 75

Figure 5.1(d) The changes of $E^*[I_t]$ (solid line) and $E[I_t]$ (dashed line) with $t$ for male (left panel) and female (right panel) populations of age 80
Figure 5.1(e) The changes of $E^*[I_t]$ (solid line) and $E[I_t]$ (dashed line) with $t$ for male (left panel) and female (right panel) populations of age 85

Figure 5.1(f) The changes of $E^*[I_t]$ (solid line) and $E[I_t]$ (dashed line) with $t$ for male (left panel) and female (right panel) populations of age 90
5.5 Bayesian pricing of the non-recourse provision

Modelling house price index

Let \( H_t \) be the house price at time \( t \) and set \( x_t = \ln H_t \). Then we assume \( x_t \) follows a local level model:  
\[
x_t = \mu_t + v_t, \quad v_t \sim \text{N}(0, V), \quad \mu_t = \mu_{t-1} + w_t, \quad w_t \sim \text{N}(0, W).
\]

The variance parameters \( V \) and \( W \) are assumed to follow inverse gamma priors as \( V \sim \text{IG}(\alpha_v, \beta_v) \) and \( W \sim \text{IG}(\alpha_w, \beta_w) \).

Risk-neutral predictive distribution

After we obtain the predictive distribution of \( x_t \) we can change it to the risk-neutral form exactly the same way as in Section 5.4. We generate \( N \) paths of MCMC sampling and denote them as \( \{x^{(j)}_t\} = \{x_1^{(j)}, x_2^{(j)}, \ldots, x_T^{(j)}\}, j = 1, 2, \ldots, N \}. \) We set \( H_t^{(j)} = \exp(x_t^{(j)}) \) for \( t = 1, 2, \ldots, T \). For a probability \( \pi^* \) over \( \{H_t^{(j)}, j = 1, 2, \ldots, N\} \) to be risk-neutral, it must satisfy  
\[
d(t) \sum_{j=1}^{N} H_t^{(j)} \pi_j^* = H_0.
\]

The risk-neutral distribution \( \pi^* \) should minimise \( \sum_{j=1}^{N} \pi_j^* \ln(\pi_j^*/\pi_j) \), subject to \( \sum_{j=1}^{N} \pi_j^* = 1 \) and the house price condition above. The solution is:  
\[
\pi_j^* = \frac{\pi_j \exp(\gamma_t H_t^{(j)})}{\sum_{j=1}^{N} \pi_j \exp(\gamma_t H_t^{(j)})} \quad \text{for} \quad j = 1, 2, \ldots, N,
\]
in which \( \gamma_t \) is computed from:  
\[
H_0 = d(t) \frac{\sum_{j=1}^{N} H_t^{(j)} \exp(\gamma_t H_t^{(j)})}{\sum_{j=1}^{N} \exp(\gamma_t H_t^{(j)})}.
\]
House prices in Japan

We use the Japanese version of the S&P / Case-Shiller Home Price Indices as an indicator for the value of the mortgaged property. The data contain monthly prices from June 1993 through December 2008. Figure 5.2 shows the changes of the indices for Japan and the US.

**Figure 5.2** S&P / Case-Shiller Home Price Indices for Japan (solid line) and the US (dashed line)

We fitted a local level model to the indices and ran 15,000 steps of MCMC simulation. We then discarded the first 5,000 steps and used the remaining 10,000 steps. The hyperparameters are set as $\alpha_x = 0.001$, $\beta_x = 0.001$, $\alpha_\mu = 0.1$, and $\beta_\mu = 0.01$. The MCMC results for $V$ and $W$ are summarised in Table 5.1. The table gives the posterior means, the posterior standard deviations, and the 95% highest posterior density (HPD) intervals for each of $V$ and $W$. The fifth column of the table lists the $p$ values of the Geweke convergence test. The $p$ values are all fairly large and seem to support the null hypothesis of convergence.
Table 5.1  Summary statistics of posterior distributions for $V$ and $W$

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>standard deviation</th>
<th>95% HPD intervals</th>
<th>$p$ value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>0.000048</td>
<td>0.000008</td>
<td>(0.000033, 0.000065)</td>
<td>0.54</td>
</tr>
<tr>
<td>$W$</td>
<td>0.000247</td>
<td>0.000028</td>
<td>(0.000194, 0.000303)</td>
<td>0.77</td>
</tr>
</tbody>
</table>

Figure 5.3 displays the MCMC sequences of $V$ and $W$. It may confirm stable convergence of each sequence. The sample autocorrelation functions for each sequence shown in Figure 5.4 suggest that samplings were performed efficiently. The posterior distributions of $V$ and $W$ depicted in Figure 5.5 all look fairly normal. Thus we may conclude that convergence is reached in each case.

Figure 5.3  MCMC sequences of $V$ and $W$
Figure 5.4  Sample autocorrelations of $V$ and $W$

![Sample autocorrelations of $V$ and $W$]

Figure 5.5  Posterior densities of $V$ and $W$

![Posterior densities of $V$ and $W$]
Evaluation of $E^*[\max(L_t - H_t, 0)]$

We set up the inputs as $H_0 = 40$ (million yen), $L_0 = 20$ (million yen), $L_t = L_0 e^{ut}$, $r = 0.015$, and $d(t) = (1 + r)^{-t}$. The results for $u = 0.04, 0.05, 0.06$ are shown in Figure 5.6.

**Figure 5.6** The changes of $E^*[\max(L_t - H_t, 0)]$ (solid line) and $E[\max(L_t - H_t, 0)]$ (dashed line) with $t$
5.6 Pricing reverse mortgages in Japan

We now combine the results in Sections 5.4 and 5.5 and set $\omega = 98$. Table 5.2 gives the values of the reverse mortgages for $u = 0.04, 0.05, 0.06$ and for $x = 65, 70, 75, 80$. We notice the following:

1. The present value of the reverse mortgage for the male cohort is larger than that for the corresponding female cohort.
2. For the male population, the present value of the reverse mortgage decreases with $x$ for each $u$.
3. For either the male or the female population, the present value of the reverse mortgage increases with $u$ for each $x$.

With $L_0 = 20$, which corresponds to half of the present house price $H_0 = 40$, most of the present values below surpass $L_0 = 20$, which suggests a promising future for reverse mortgages in Japan.

Table 5.2 The present values of the reverse mortgages

\begin{tabular}{|c|c|c|}
\hline
\textbf{x} & \textbf{male} & \textbf{female} \\
\hline
65 & 27.60459 & 20.18924 \\
70 & 25.34564 & 19.12708 \\
75 & 24.26313 & 20.07780 \\
80 & 21.93222 & 19.11508 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|}
\hline
\textbf{x} & \textbf{male} & \textbf{female} \\
\hline
65 & 30.56589 & 22.32114 \\
70 & 28.13248 & 21.30440 \\
75 & 26.88519 & 22.40625 \\
80 & 23.85832 & 21.01858 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|}
\hline
\textbf{x} & \textbf{male} & \textbf{female} \\
\hline
65 & 32.43857 & 23.54640 \\
70 & 30.10885 & 22.71018 \\
75 & 29.03342 & 24.18518 \\
80 & 25.70015 & 22.78073 \\
\hline
\end{tabular}
5.7 Concluding remarks

In this section, we have proposed a Bayesian multivariate framework to price reverse mortgages and applied it to Japanese data. The results indicate a promising future for reverse mortgages in Japan.
6 References


Chi I. 2004. Retirement income protection in Hong Kong. International Conference on
Pensions in Asia: Incentives, Compliance, and Their Role in Retirement, Tokyo,
Japan.
option for ‘asset-rich and cash-poor’ Singaporeans. Working Paper, Department of
Economics, National University of Singapore.
Coale A. and Guo G. 1989. Revised regional model life tables at very low levels of mortality.
Continuous Mortality Investigation (CMI). 1999. Report 17, Institute of Actuaries and
Faculty of Actuaries.
Continuous Mortality Investigation (CMI). 2009. Report 23, Institute of Actuaries and
Faculty of Actuaries.
Coughlan G. 2009. Longevity risk transfer: indices and capital market solutions. In Barrieu
Crawford T., De Haan R., and Runchey C. 2008. Longevity risk quantification and
management: a review of relevant literature. Society of Actuaries.
University.


Directorate General of Budget, Accounting, and Statistics (DGBAS), Executive Yuan, Taiwan. 2010. Table 11 Distribution of income recipients by age.


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Human Mortality Database (HMD). 2012. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). www.mortality.org


Koissi M.C. and Shapiro A.F. 2008. The Lee-Carter model under the condition of variables age-specific parameters. 43rd Actuarial Research Conference, Regina, Canada.


Office of the Commissioner of Insurance, Hong Kong. 2010. Table L7 Annuity and other in-force business / individual annuity new business.
Verlag.
linear modelling approach for England and Wales mortality projections. *Applied
Renshaw A.E. and Haberman S. 2006. A cohort-based extension to the Lee-Carter model for
Renshaw A.E. and Haberman S. 2008. On simulation-based approaches to risk measurement
in mortality with specific reference to Poisson Lee-Carter modelling. *Insurance:
Mathematics and Economics* 42: 797-816.
Russolillo M., Giordano G., and Haberman S. 2011. Extending the Lee-Carter model: a three-
and life expectancy: a comparison of ten principal component methods. *Demographic
for mortality projections, with applications to immediate annuitants’ and life office
Social Security Administration, U.S. 2011. Social security programs throughout the world:
Asia and the Pacific, 2010.
www.mrc-bsu.cam.ac.uk/bugs/
of Finance* 51: 1633-1652.
Paper* PI-1004.
Actuarial Science* 5: 143-162.


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