Longevity Risk Modelling and Management via Securities Linked to Parametric Mortality Indices

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# Table of Contents

1. **Modelling with Causes of Deaths**
   - 1.1 Mortality patterns by causes of deaths
   - 1.2 Modelling mortality by causes of deaths
   - 1.3 Further remarks

2. **Longevity Basis Risk and Impact of Model Uncertainty**
   - 2.1 Assessment of longevity basis risk
   - 2.2 Mortality modelling and simulations
   - 2.3 Effectiveness of index-based hedging
   - 2.4 Further remarks

3. **Parametric Mortality Indices and Out-of-the-Money Longevity Hedges**
   - 3.1 Parametric mortality indices
   - 3.2 CBD mortality indices
   - 3.3 Risk-neutral CBD model
   - 3.4 Hedging instruments
   - 3.5 Longevity risk exposures
   - 3.6 Developing longevity hedges
   - 3.7 Numerical illustrations
   - 3.8 Further remarks
   - 3.9 Derivation of formulas

4. **References**

5. **Acknowledgements**
1 Modelling with Causes of Deaths

While most mortality projection work in the actuarial literature deals with all-causes mortality data, modelling mortality with cause-specific data can deliver additional insights. Firstly, a more detailed investigation of the major causes of deaths and their trends can provide supplementary information on forecasting the total deaths. Historically, each cause of death has its own unique patterns, and it would be wasteful to ignore such information totally and disregard the otherwise hidden features. Secondly, health insurers, health policymakers, and medical researchers would be interested in cause-specific mortality rates. Studying these data would help them calculate more accurate prices and reserves, direct appropriate resources to population subgroups, and understand better the biological mechanisms that lead to different diseases. Moreover, data collection for many populations has improved significantly over the recent decades, which greatly facilitates various investigation and modelling exercises using cause-specific mortality data.

The historical cause-specific mortality patterns of Hong Kong, Japan, Australia, and New Zealand are examined in Section 1.1. Mortality modelling and forecasting by causes of deaths are discussed in Section 1.2. Further comments are given in Section 1.3.
1.1 Mortality patterns by causes of deaths

Mortality data for four populations in the Asia-Pacific region are collected from the Human Mortality Database (HMD 2017), World Health Organization (WHO), and Census and Statistics Department of Hong Kong. The data from the WHO Mortality Database are further broken down by causes of deaths. The classification of the specific causes follows the International Classification of Diseases (ICD), a global health care diagnostic standard maintained by the WHO. The data collected are from 1980 onwards and in five-year age groups. Accounting for such issues as data availability for a sufficient period of time, comparability between versions of ICD, as well as geographical and socio-economic relationships, slightly more than thirty years of cause-specific mortality data of Hong Kong, Japan, Australia, and New Zealand are collected for our analysis. We focus on the five major causes of deaths (cancer, infectious and parasitic diseases, circulatory diseases, respiratory diseases, external causes), which explain more than 80% of the deaths in the data.

Figure 1.1 plots the cause-specific log central death rates of females and males for each population. The overall mortality improvement over time is clear to see. For each cause of deaths, the age profiles are all quite smooth, which means that the lives at adjacent age groups tend to have similar mortality levels from that cause. The improvements in mortality related to circulatory diseases have been prominent across the populations. For cancer and infectious diseases, however, there has often been an increase in their mortality levels at ages 85 and above. This observation may be explained by the fact that as the expected lifespan is prolonged but the maximum lifespan is unchanged (i.e. rectangularisation of survival curve), more and more deaths concentrate after age 85, during which the human ageing process comes to an end and the biological mechanism deteriorates rapidly.

There are also country-specific patterns being observed. For Hong Kong, mortality related to cancer and infectious diseases at the oldest age group has decreased in the recent two decades, and the decline in mortality from external causes has remained strong over time. For Australia, there has been an increase in mortality related to infectious diseases for the whole age range, as well as external and other causes for ages 85 and above. For New Zealand, mortality from other causes has increased at ages 85 and above. For both Australia and New Zealand, mortality from external causes does not really increase with age until about 70.
Figure 1.1 Cause-specific log central death rates

Hong Kong
Figure 1.1  Continued

Japan
Figure 1.1  Continued

Australia
Figure 1.1 Continued

New Zealand
1.2 Modelling mortality by causes of deaths

In this section, we apply the Lee and Carter (1992) model to the cause-specific death rates and also the all-causes death rates, and compare the differences in the corresponding results. The Lee-Carter model is the most popular method for projecting future death rates. The basic model structure is $\ln m_{x,t} = a_x + b_x k_t$, in which $m_{x,t}$ is the central death rate at age $x$ in year $t$, $a_x$ describes the overall age profile, and $b_x$ measures the sensitivity of the log death rate to changes in the mortality index $k_t$, subject to two constraints $\sum_x b_x = 1$ and $\sum_t k_t = 0$.

We use the Poisson distribution and maximum likelihood to estimate the parameters. This model has a simple structure and often produces a highly linear (decreasing) time series of $k_t$ for many countries’ mortality data. The mortality index is then assumed to follow a random walk with drift as $k_t = \mu + k_{t-1} + \epsilon_t$, where $\mu$ is the drift term and $\epsilon_t$’s are independent, identically distributed error terms with mean zero and variance $\sigma^2$.

Figure 1.2 shows the Lee-Carter parameter estimates for each cause of deaths. The smooth age profiles of $a_x$ are in line with those in Figure 1.1, in which the cause-specific mortality levels increase over age. The decreasing trends of the mortality indices $k_t$ are fairly linear over time, while some notable exceptions can be observed. For instance, the two spikes regarding external causes in Japan occur in 1995 (Kobe earthquake) and 2011 (Tohoku earthquake and tsunami). For other causes in Japan, in the most recent decade, there is an increase in the mortality index instead of the usual decline. This sudden reversal requires further investigation as the other causes category is heterogenous and involves various other causes of deaths. For infectious diseases in Australia, the mortality index is increasing, unlike all the others, which may be related to the blood poisoning (sepsis) issue in that country. For infectious diseases in New Zealand, the mortality index has also reversed its trend after 2000.

The sensitivity $b_x$ often becomes very small beyond age 80, which broadly means that the medical advancement to date has limited impact on the oldest lives. Interestingly, there are distinct sensitivity patterns between countries, sexes, and causes of deaths:

- Considering Hong Kong, for cancer and other causes, the sensitivity tends to decrease with age; for circulatory diseases and external causes, the sensitivity peaks at about ages 60 to 70; for respiratory and infectious diseases, the sensitivity starts to drop
after around age 60.

- Considering Japan females, for cancer and respiratory diseases, the sensitivity is quite even amongst those aged 40 to 80 and drops rapidly at older ages; for circulatory diseases, external causes, and other causes, the sensitivity is highest at around age 70; for infectious diseases, the sensitivity decreases smoothly with age.

- Considering Japan males, for respiratory and infectious diseases, the sensitivity is quite stable among those aged 70 and below and drops rapidly at older ages; for circulatory diseases and other causes, the sensitivity peaks at around age 80; for cancer and external causes, the sensitivity tends to decrease with age.

- Considering Australia, for cancer and other causes, the sensitivity tends to decrease with age; for circulatory diseases, respiratory diseases, and external causes, the sensitivity is highest at about age 60; for infectious diseases, the sensitivity is higher at both the younger and the older ends.

- Considering New Zealand, for cancer and other causes, the sensitivity tends to decline with age; for circulatory and respiratory diseases, the sensitivity peaks at about age 60 or below; for infectious diseases and external causes, the sensitivity increases roughly with age.

Figure 1.3 compares the projected log central death rates between aggregating the separate projections of different causes of deaths and applying the Lee-Carter model directly to the all-causes aggregate data. The projected death rates in 2030 from the former approach are only slightly higher than those from the latter approach. But in 2050, the differences are clearly larger, particularly at older ages. This discrepancy may arise partially from projecting the increase in cancer-caused mortality at ages 85 and above. It may also be caused by projecting the relatively slow improvement in cancer-caused mortality (compared to circulatory and respiratory diseases). Moreover, in the projections, the importance of cancer-caused mortality grows with time, while circulatory diseases reduce in significance. Currently, deaths caused by cancer account for about 40% on average of the total deaths of ages 40 and above, and deaths from circulatory diseases contribute to around 20%. For the projected values in 2050, deaths caused by cancer increase by 10% to 20% of the total, whilst deaths from circulatory diseases drop by about the same magnitude. Comparatively, the proportions of the other causes of deaths are more stable in the projections.
Figure 1.2  Lee-Carter parameter estimates ($a_x, b_x, k_t$) (Hong Kong females)

cancer

circulatory diseases

respiratory diseases

infectious diseases

external causes

other causes
Figure 1.2  Continued (Hong Kong males)

cancer

circulatory diseases

respiratory diseases

infectious diseases

external causes

other causes
Figure 1.2  Continued (Japan females)

- cancer
- circulatory diseases
- respiratory diseases
- infectious diseases
- external causes
- other causes
**Figure 1.2  Continued (Japan males)**

- cancer
- circulatory diseases
- respiratory diseases
- infectious diseases
- external causes
- other causes
**Figure 1.2** Continued (Australia females)

- **cancer**
- **circulatory diseases**
- **respiratory diseases**
- **infectious diseases**
- **external causes**
- **other causes**
Figure 1.2  Continued  (Australia males)

- cancer
- circulatory diseases
- respiratory diseases
- infectious diseases
- external causes
- other causes
**Figure 1.2 Continued** (New Zealand females)

- **Cancer**
- **Circulatory diseases**
- **Respiratory diseases**
- **Infectious diseases**
- **External causes**
- **Other causes**
Figure 1.2  Continued (New Zealand males)

cancer

circulatory diseases

respiratory diseases

infectious diseases

external causes

other causes
Figure 1.3  Projected log central death rates (Hong Kong)
Figure 1.3  Continued (Japan)
Figure 1.3  Continued (Australia)
Figure 1.3  Continued (New Zealand)
1.3 Further remarks

From the analysis above, it can clearly be seen that cause-specific data can provide useful information for understanding the underlying mortality patterns. The relative mortality changes between different causes of deaths decide the aggregate mortality levels. While modelling aggregate data only would miss out these important details, modelling cause-specific data also has its own problems. First, aggregate data are generally more available than cause-specific data, which limit the extent of possible investigation. But as data collection continues to improve for many populations, we can foresee that this problem will be overcome gradually. Second, projecting separately by each cause of deaths has a tendency to produce higher overall death rates, because of the specific trends of certain causes of deaths and also the domination of the most slowly declining cause of deaths in the long term. Particularly, cures may be invented in the future to treat once an incurable disease. To more properly project these trends, further research is required on the interplays between the current main causes of deaths, as well as the emergence of new major causes of deaths. Nevertheless, there will always be a degree of uncertainty in the projections. Medical scientists are working endlessly to pursue a cure for the major diseases and even the ageing process itself. Time will tell whether and by what extent they can achieve that.
2 Longevity Basis Risk and Impact of Model Uncertainty

In this section, we investigate the impact of model uncertainty on hedging longevity risk with index-based derivatives and assessing longevity basis risk, which arises from the mismatch between the hedging instruments and the portfolio being hedged. We apply the bivariate Lee-Carter model, the common factor model, and the M7-M5 model, with separate cohort effects between the two populations, and various time series processes and simulation methods, to build index-based longevity hedges and measure the hedge effectiveness. Based on our modelling and simulations from hypothetical scenarios, the estimated levels of hedge effectiveness are around 50% to 80% for a large pension plan, and the model selection plays a very important role in the estimation. We also experiment with a modified bootstrapping approach to incorporate the uncertainty of model selection into the modelling of longevity basis risk. The hedging results under this approach may approximately be seen as a ‘weighted’ average of those calculated from the different model candidates.

The concepts of index-based hedging and longevity basis risk are introduced in Section 2.1. A list of two-population mortality projection models, time series processes, and simulation approaches selected for our analysis are given in Section 2.2. The levels of hedge effectiveness under different model settings and assumptions and simulated scenarios are examined in Section 2.3, based on which the impact of model uncertainty is discussed. Further remarks and suggestions for future research are set out in Section 2.4.
2.1 Assessment of longevity basis risk

Continual increase in longevity worldwide remains a serious concern for pension plan sponsors and annuity providers. The major issue is the presence of longevity risk, which is the risk of paying more than expected because of unanticipated mortality improvements. Broadly speaking, there are three ways to mitigate longevity risk (e.g. Cairns et al. 2008). The first is to transfer the unwanted risk to an insurer or reinsurer by paying a premium. The problem of this usual approach is that insurers and reinsurers may also have limited appetite, and that a failure of a major player could cause disastrous systemic outcomes. The second way is natural hedging, which makes use of the opposite movements between the values of annuities and life insurances arising from changes in mortality levels (e.g. Li and Haberman 2015). While certain large institutions may have the resources and economies of scale to offer both lines of products and so exploit this hedging effect, many other financial entities do not have such necessary conditions to follow suit.

The third way is the use of capital market solutions, such as insurance securitisation (e.g. Cowley and Cummins 2005), and mortality- and longevity-linked securities (e.g. LCP 2012, Coughlan et al. 2007). The former involves packaging insurance and business risks into securities, such as bonds with coupons and principal payments depending on the performance of the underlying portfolio. The latter has two types of transactions, bespoke and index-based. Bespoke transactions are tailored to individual circumstances, for example, pension buy-ins, buy-outs, and longevity swaps. In contrast, index-based solutions are constructed such that the cashflows are linked to the values of selected mortality indices. As noted in Zhou and Li (2017a), there is an imbalance between demand and supply in longevity risk transfer. The insurance industry alone cannot generate sufficient supply for accepting the risk due to capital constraints. Accordingly, an important, recent idea is to design standardised products based on well-specified mortality indices, in order to draw more investors’ interest and develop market liquidity.

One significant challenge in implementing index-based hedging in practice is the existence of longevity basis risk. That is, there is a mismatch or discrepancy between the hedging instruments (linked to a reference population) and the pension or annuity portfolio being hedged (with the underlying book population). Haberman et al. (2014) established a mortality modelling framework incorporating three fundamental sources of longevity basis
risk. They include *demographic* basis risk which comes from demographic or socioeconomic differences between the book and reference populations, *sampling* basis risk due to the randomness of outcomes of individual lives, and *structural* basis risk in terms of how the payoff structures differ between the hedging instruments and the portfolio to be hedged. Li et al. (2017) then adopted the framework and focused on assessing longevity basis risk in realistic scenarios under practical circumstances. The major finding is that for a large portfolio, about 50% to 80% of its longevity risk can often be reduced via index-based longevity swaps, while for a small portfolio, the risk reduction is usually less than 50%.

In particular, from a modelling viewpoint, or a regulatory perspective such as Solvency II (CEIOPS 2010), demographic basis risk can further be divided into process error, parameter error, and model error (or process risk, parameter uncertainty, model uncertainty, respectively). Process error refers to the variability in the time series, parameter error arises from the uncertainty in estimating model parameters, and model error reflects the uncertainty in model selection. Figure 1 below provides a graphical summary of all these risk components. While a number of authors proposed different ways to allow for both process error and parameter error in index-based hedging (e.g. Li and Luo 2012, Cairns 2013, Tan et al. 2014, Haberman et al. 2014), relatively little attention has been given to assessing model error. Most of the studies covered process error only or considered both process error and parameter error, while some compared the hedging results briefly between a few different mortality projection models. Li et al. (2016) took a step further by examining the effectiveness of a hedging strategy based on a particular model but under a simulated environment produced from another model, and found that coherence is a critical assumption in measuring the hedge effectiveness. Li et al. (2017) conducted a detailed sensitivity analysis by changing various modelling assumptions. They noted that the most important ones were the coherence property, behaviour of simulated future variability, simulation method, and additional model features such as structural mortality changes.

In the following sections, we perform a more elaborate investigation on model error in an attempt to supplement the current literature from two perspectives. Firstly, we assess the hedge effectiveness based on three broad ‘families’ of mortality projection models, a range of time series processes under each model, and a number of simulation methods for each combination. Secondly, we adopt the modified semi-parametric bootstrapping approach in Yang et al. (2015) and incorporate all the three errors in an integrated manner. From these
two different perspectives, we attempt to obtain better insights into the potential impact of selecting an inappropriate model in hedging longevity risk with index-based solutions.

**Figure 2.1  A modelling framework for longevity basis risk**
2.2 Mortality modelling and simulations

Recently, there is an emerging branch of literature that focuses on modelling multiple populations jointly. Haberman et al. (2014) and Li et al. (2014) provide a comprehensive review of several multi-population mortality projection models for measuring longevity basis risk. While there are probably more than thirty models proposed in the literature, we select the ones below for our analysis, as they can be deemed as ‘key representatives’ of the major ‘families’ of mortality projection models.

The first belongs to the Lee and Carter (1992) family, in which there are some possible options (1a, 1b, 1c, or 1d) for modelling the time-varying parameters:

\[
\ln m_{x,t,i} = \alpha_{x,i} + \beta_{x,i} \kappa_{t,i} + t_{-x,i} \tag{Lee-Carter model with cohort} \quad (1)
\]

with

\[
K_t = \Theta + K_{t-1} + \Delta_t \tag{bivariate random walk with drift} \quad (1a)
\]

or

\[
\kappa_{t,1} = \theta + \kappa_{t-1,1} + \delta_t \tag{random walk with drift} \quad (1b)
\]

or

\[
K_t - K_{t-1} = \Theta + \sum_{i=1}^{p-1} \Gamma_i (K_{t-i} - K_{t-i-1}) + \Delta_t \tag{VECM( p )} \quad (1c)
\]

or

\[
K_t - K_{t-1} = \Theta + \sum_{i=1}^{p} \Gamma_i (K_{t-i} - K_{t-i-1}) + \Delta_t \tag{VARIMA( p , 1, 0)} \quad (1d)
\]

The term \( m_{x,t,i} \) is the central death rate at age \( x \) in year \( t \) of population \( i \), in which \( i=1 \) refers to the reference population and \( i=2 \) refers to the book population. The parameter \( \alpha_{x,i} \) is the general age schedule, \( \kappa_{t,i} \) is the mortality index over time, \( K_t = (\kappa_{t,1}, \kappa_{t,2})' \), and \( \beta_{x,i} \) is the age sensitivity of the log death rate to the mortality index. The cohort parameter \( t_{-x,i} \) is added when there are significant patterns in the residuals against cohort year (e.g. Haberman and Renshaw 2011). It can be modelled with an autoregressive integrated moving average (ARIMA) process for future projections and simulations, though this is not required in the analysis in the next section. Following Cairns et al. (2009), the first and last five cohorts are
excluded from the fitting procedure due to a lack of data.

The first method suggested in Carter and Lee (1992) is fitting the Lee-Carter model to each population separately and then modelling the relationship between the two resulting mortality indices. This approach involves the use of a bivariate random walk with drift (1a) naturally, in which $\Theta$ is the vector drift term and $\Delta$ is the Gaussian vector error term. By contrast, the third method in Carter and Lee (1992) is treating the two mortality indices as a co-integrated process (1b), where $\theta$, $a_0$, and $a_i$ are the parameters of the process, and $\delta_i$ and $\omega_i$ are independent Gaussian error terms. Moreover, Yang and Wang (2013) proposed using a vector error correction model (VECM) of order $p$ (1c), in which $\Theta$ is the vector constant term, $\Pi$ and $\Gamma$, are the matrix components, and $\Lambda_i$ is the Gaussian vector error term. Finally, we also consider a vector autoregressive integrated moving average (VARIMA) process (1d), which has been explored by Chan et al. (2014) in modelling other kinds of parametric mortality indices. The order chosen is $(p, 1, 0)$, $\Theta$ is the vector constant term, $\Gamma$, is the autoregressive matrix, and $\Lambda_i$ is the Gaussian vector error term. Note that all the options (1a) to (1d) would generally lead to non-coherence in mortality projections, i.e. the ratio of projected death rates (central estimates) between the two populations at each age group does not converge to a constant over time (e.g. Cairns et al. 2011).

The second is the Li and Lee (2005) family, which is an extension of the Lee-Carter model and assumes a common factor between the two populations. Again, there are a few possible choices (2a, 2b, 2c, or 2d) in modelling the temporal parameters:

$$\ln m_{s,t,j} = \alpha_{s,j} + \beta_s^\lambda \kappa_{s}^\lambda + \sum_{j=1}^{n} \beta_{s,i,j} \kappa_{s,i,j} + t_{s-x,i,j} \quad \text{(generalised common factor model with cohort)}$$

$$\kappa_{s}^\lambda = d + \kappa_{s-1}^\lambda + \varepsilon_t$$

(random walk with drift) (2)

with

$$\kappa_{s,i,j} = \theta_{s,j} + \gamma_{1,i,j} \kappa_{s-1,i,j} + \delta_{s,i,j} \quad \text{(AR(1); independent } \delta_{s,1,j} \text{ and } \delta_{s,2,j})$$

(2a)

or

$$\kappa_{s,i,j} = \theta_{s,j} + \sum_{r=1}^{p} \gamma_{r,i,j} \kappa_{s-r,i,j} + \delta_{s,i,j} \quad \text{(AR( } p \text{ ); independent } \delta_{s,1,j} \text{ and } \delta_{s,2,j})$$

(2b)

or
\[ \kappa_{r,i,j} = \theta_{r,i,j} + \sum_{r=1}^{p} \gamma_{r,i,j} \kappa_{r-i,j} + \delta_{r,j} \]  
\text{ (AR( } p \text{ ); correlated } \delta_{1,j} \text{ and } \delta_{2,j} \text{) \hspace{1cm} (2c)}

or

\[ K_{t,i} = \Theta_{j} + \sum_{r=1}^{p} \Gamma_{r,j} K_{t-r,j} + \Delta_{t,j} \]  
\text{ (VAR( } p \text{ ) \hspace{1cm} (2d)}

The parameter \( \alpha_{i,j} \) is the general age schedule of population \( i \), \( \kappa^i \) is the mortality index of the common factor for both populations with age sensitivity \( \beta^i \), and \( \kappa_{r,i,j} \) is the time component of the \( j \)th additional factor for population \( i \) with age sensitivity \( \beta_{r,i,j} \). The common mortality index is modelled as a random walk with drift as usual, with \( d \) as the drift and \( \varepsilon_t \) as the Gaussian error. Originally, Li and Lee (2005) used only one additional factor; later, Li (2013) proposed using \( n \) additional factors where necessary, and Yang et al. (2016) further suggested adding cohort parameters such as \( t_{t-i,j} \).

Li and Lee (2005) assumed that \( \kappa_{r,i,1} \) followed an autoregressive (AR) process of order one, and that the error terms are independent between the populations \( (2a) \). On the other hand, Li (2013) tested a more general AR( \( p \) ) process for each \( \kappa_{r,i,j} \) \( (2b) \), and Zhou and Li (2017a) assumed that the error terms \( \delta_{r,i,1} \)'s are correlated between the populations \( (2c) \).

The parameters \( \theta_{r,i,j} \) and \( \gamma_{r,i,j} \) are the constant and autoregressive terms respectively. Lastly, we also include a vector autoregressive (VAR) process of order \( p \) for \( K_{t,i} = \left( \kappa_{r,i,1}, \kappa_{r,i,2} \right) \) \( (2d) \), which has been adopted by Haberman et al. (2014) for modelling some other mortality indices. The notation \( \Theta_{j} \) is the vector constant term, \( \Gamma_{r,j} \) is the autoregressive matrix, and \( \Delta_{t,j} \) is the Gaussian vector error term. All the choices \( (2a) \) to \( (2d) \) can result in coherence in mortality projections, i.e. the ratio of projected death rates between the two populations at each age group tends to a constant over time, if the selected time series processes are weakly stationary. More technical details of time series modelling can be found in Tsay (2002).

The final one is an extension of the CBD model by Cairns et al. (2006b) and is proposed by Haberman et al. (2014), being referred to as the M7-M5 model. There are also a number of possible time series processes to choose from \( (3a, 3b, 3c, \text{ or } 3d) \) under this model:

\[ \logit q_{x,t}^R = \kappa_{1,t}^R + (x - \bar{x}) \kappa_{2,t}^R + \left( (x - \bar{x})^2 - \sigma^2 \right) \kappa_{3,t}^R + t_{t-x}^R \]  
\text{ (extended CBD model with cohort)
logit \( q_{x,t}^B - \text{logit } q_{x,t}^R = \kappa_{t,1}^B + (x - \bar{x})\kappa_{t,2}^B + \iota_{t-x}^B \) (CBD model structure with cohort)  

\[ \text{with} \]

\[ K_t^R = \Theta^R + K_{t-1}^R + \Delta_t^R \]  

(multivariate random walk with drift)  

\[ K_t^B = \Theta^B + \sum_{r=1}^{p} \Gamma_r K_{t-r}^R + \Delta_t^B \]  

(VAR(\( p \)); independent \( \Delta_t^R \) and \( \Delta_t^B \))  

\[ \text{or} \]

\[ K_t^R - K_{t-1}^R = \Theta^R + \sum_{r=1}^{p} \Gamma_r (K_{t-r}^R - K_{t-r-1}^R) + \Delta_t^R \]  

(VARIMA(\( p \), 1, 0))  

\[ K_t^B = \Theta^B + \sum_{r=1}^{p} \Gamma_r K_{t-r}^B + \Delta_t^B \]  

(VAR(\( p \)); independent \( \Delta_t^R \) and \( \Delta_t^B \))  

\[ \text{or} \]

\[ K_t^R = \Theta^R + K_{t-1}^R + \Delta_t^R \]  

(multivariate random walk with drift)  

\[ K_t^B = K_{t-1}^B + \Delta_t^B \]  

(bivariate random walk without drift; independent \( \Delta_t^R \) and \( \Delta_t^B \))  

\[ \text{or} \]

\[ K_t^R = \Theta^R + K_{t-1}^R + \Delta_t^R \]  

(multivariate random walk with drift)  

\[ K_t^B = \Theta^B + \sum_{r=1}^{p} \Gamma_r K_{t-r}^B + \Delta_t^B \]  

(VAR(\( p \)); correlated \( \Delta_t^R \) and \( \Delta_t^B \))  

The term \( q_{x,t}^R \) is the mortality rate at age \( x \) in year \( t \) of the reference population, \( \kappa_{t,1}^R \), \( \kappa_{t,2}^R \), and \( \kappa_{t,3}^R \) represent the level, slope, and curvature of the mortality curve in year \( t \) (e.g. Cairns et al. 2009), and \( \iota_{t-x}^R \) is the cohort parameter. The difference in the logit mortality rate between the book and reference populations (i.e. \( \text{logit } q_{x,t}^B - \text{logit } q_{x,t}^R \)) is then modelled with another CBD structure of \( \kappa_{t,1}^B \) and \( \kappa_{t,2}^B \), which explain the differences between the two populations, together with another cohort parameter \( \iota_{t-x}^B \). Note that the original M7-M5 model has a common cohort parameter for both populations, while we use a separate cohort parameter for each population in order to allow for the different cohort effects in our data. The notation \( \bar{x} \) is the average age of the age range in the data, and \( \sigma^2 \) is the average value of \( (x - \bar{x})^2 \) of the age range.
In contrast to families (1) and (2) above, where both populations are modelled in parallel, Haberman et al. (2014) fitted the reference population first and then the differences between the two populations. One major argument for this treatment is that there are usually much more data available for the reference population than for the book, and so it would be more appropriate to base the main trends on the reference population and consider the differences of the book population afterwards.

Haberman et al. (2014) assumed that \( K^R_t = (\kappa^R_{t,1}, \kappa^R_{t,2}, \kappa^R_{t,3})' \) follows a multivariate random walk with drift, \( K^B_t = (\kappa^B_{t,1}, \kappa^B_{t,2})' \) follows a VAR(1) process, and \( \Delta^R_t \) and \( \Delta^B_t \) are independent, whereas we adopt a more general VAR(\( p \)) process here for \( K^R_t \) (3a). Accordingly, we also consider three other alternatives. First, similar to replacing the option (1a) with (1d), we replace the multivariate random walk with drift with a VARIMA(\( p, 1, 0 \)) process for \( K^R_t \) (3b). Second, we replace the VAR(\( p \)) process with a bivariate random walk without drift for \( K^B_t \) (3c), as we have seen in our analysis that the calculated \( \kappa^B_{t,1} \) and \( \kappa^B_{t,2} \) fluctuate around roughly a constant level in many cases. Moreover, in line with the options (2b) and (2c), we test the correlation assumption between \( \Delta^R_t \) and \( \Delta^B_t \) (3d). All the vectors and matrices of parameters and error terms have similar meanings as previously. Note that the options (3a), (3b), and (3d) can generate coherence in the projected mortality rates approximately if the selected VAR process is weakly stationary. The option (3c) is always coherent.

Besides the central estimates’ coherence, the simulated future variability is also a significant feature, as the relative potential movements between the book mortality and reference mortality is one main driver of longevity basis risk. This issue, however, has largely been overlooked in the current literature, which usually focuses on the mortality projection models and the (weak) stationarity of the time series processes, but not on the simulated future variability resulting from time series modelling. Based on the M7-M5 model and the CAE+Cohorts model (which is an extension of the Lee-Carter model), Li et al. (2017) found that the behaviour of simulated future variability of the ‘book minus reference’ component is the most important time series modelling assumption. It should be noted that while the random walk and integrated autoregressive processes above produce increasing variability over time in the simulations, the autoregressive processes (not integrated ones) generate
bounded variability instead. The combined or alternative use of these time series processes would give rise to varying hedging results, as shown in the next section.

In addition to the Lee-Carter, Li-Lee, and CBD families described above, there are also a number of other similar approaches for modelling longevity basis risk (e.g. Plat 2009, Coughlan et al. 2011, Ngai and Sherris 2011, Tsai et al. 2011, Cairns 2013, Li et al. 2016). Furthermore, while there is an abundant amount of time series processes developed in econometric studies, it appears that only a handful of them would actually turn out to be useful for modelling longevity basis risk in practice, due to the usually short length and small amount of annual book data being available.

Regarding the simulation methods, we consider four different choices (e.g. Li 2014). The first is simply Monte Carlo simulation as in Lee and Carter (1992), where only process error is allowed for. The second is the semi-parametric bootstrapping approach proposed in Brouhns et al. (2005), and the third is the residuals bootstrapping approach suggested by Koissi et al. (2006), both of which include process error and also parameter error. These simulation methods have initially been catered for single-population mortality projection models, and some adaptations are needed for the two-population models stated above. Particularly, in addition to the choice of resampling the reference and book residuals separately, we also group together the two populations’ residuals in each age-time cell as an individual (bivariate) data point for resampling, such that more information about the relationships between the two populations may be embedded into the simulated samples (Li and Haberman 2015).

Apart from measuring the hedge effectiveness by testing various models and investigating different simulated scenarios, we also adopt the modified semi-parametric bootstrapping approach in Yang et al. (2015) and incorporate process error, parameter error, and model error simultaneously. It is effectively an extension of the bootstrapping approach suggested by Brouhns et al. (2005). First, a pseudo sample of the number of deaths is simulated from the Poisson distribution with the observed number of deaths as the mean. Suppose there are two or more competing model candidates under consideration. Each model in turn is fitted to the pseudo data sample and the corresponding model parameters are estimated. On the basis of some pre-determined model selection criteria, the most ‘optimal’ model is selected for this particular pseudo data sample. The time series of the selected model’s temporal parameters are then simulated into the future. Lastly, future death rates are
produced from the estimated and simulated parameters of the selected model. So far in this process, the pseudo data sample is used for generating only one future scenario via one selected model. The entire process above can then be repeated iteratively to create, say, 5,000 future scenarios in total\(^1\), in which different models may be chosen in different scenarios. In this way, all the three errors are embedded in the simulated future death rates. This modified bootstrapping approach provides a different perspective to understand the impact of model uncertainty, though we acknowledge that the simulation time can be significantly lengthened, especially when more competing models and selection criteria are included in the process. Again, the bootstrapping process needs to be properly adapted to the two-population models. Figure 2.2 below gives a list of the modelling techniques we experiment with to tackle the different error components. In the next section, we will discuss in detail the similarities and differences between the results on some hypothetical hedging scenarios generated by all these modelling and simulation methods.

**Figure 2.2 Allowance for process error, parameter error, and model error**

\[^1\] The size of 5,000 iterations appears to produce fairly stable results in the next section. We find that the estimated levels of hedge effectiveness usually differ by less than 3% in magnitude between repeated runs on the same case study.
2.3 Effectiveness of index-based hedging

We have collected two sets of data for the book and reference populations. The first set is composed of the male assured lives and pensioners data (book) from the Continuous Mortality Investigation (CMI), and England and Wales male data (reference) from the Human Mortality Database (HMD 2017), for the period from 1983 to 2006. The second set comprises Australia, New Zealand, Japan, Taiwan, Hong Kong, and Singapore male data from the HMD and governmental statistics departments, for years 1980 to 2016. Since Asia-Pacific insured data are scarce, we use the data of New Zealand, Taiwan, and Singapore, which have relatively smaller sizes, as a proxy for the book population, and the data of their larger neighbours for the reference population. The age range considered is from 60 to 89. Figure 2.3 plots the log central death rates of three age groups over time. It can be seen that the mortality declining trends of different populations or regions are roughly in line with one another in the past few decades. In general, the death rates of the assured lives and pensioners are lower than those of English and Welsh population. One potential problem of the assured lives and pensioners data is that there may have been different contributors to the data over the period and there would then be some extent of heterogeneity or inconsistency. Judging simply from the graphs below, it seems that the issue is not significant. Australian death rates are slightly lower than New Zealand death rates, and Japan has experienced lower mortality levels than Taiwan. Hong Kong has had lower mortality experience than Singapore, while the latter is catching up fairly quickly in the last decade.

We mainly consider a hypothetical situation of a large pension plan with 10,000 members for a particular cohort. All the pensioners are aged 65 and every pensioner receives $1 on survival of each year in the next 25 years. Suppose that the pension plan financier attempts to minimise its longevity risk exposure by building a longevity hedge with index-based S-forwards (e.g. LLMA 2010), and that the S-forward maturing at every future age is available for the same birth cohort as the pensioners. For a floating rate receiver, the payoff on maturity of a S-forward is equal to the actual survivor index (observed on maturity) minus the forward survivor index (set at time 0), in which the survivor index is the percentage of the initial reference population who are still alive on maturity. Assume that the current forward values are equal to the central estimates, setting a zero risk premium for convenience, and that the interest rate is constant at 1% p.a. throughout the period, considering the current low
interest rate environment. The valuation date is taken as just after the end of the data period.

**Figure 2.3   Log central death rates of male lives in UK and Asia-Pacific regions**

After fitting the above-mentioned models to the data and carrying out the simulations as in the previous section, we use the simulated future death rates of the book population and the binomial distribution to further simulate the number of surviving pensioners in each future year (e.g. Haberman et al. 2014). We can then obtain random samples of the present value of the pension plan liability. On the other hand, we use the simulated future death rates of the reference population to determine the random S-forwards payoffs, which are discounted to the valuation date. The notional amounts of the S-forwards are calculated by numerical optimisation in order to maximise the level of hedge effectiveness (e.g. Li et al. 2017). The weights are mostly in the range of 0.6 to 1.1 (per person) in different cases of our analysis. Figure 2.4 demonstrates the longevity hedging scheme based on the use of index-based S-forwards. Note that the counterparties can be a financial exchange or intermediary that brings the market investors (hedge providers) and the pension plan (hedger) into conducting standardised transactions.
Table 2.1 lists the BIC (Bayesian Information Criterion) values of fitting the three two-population mortality projection models to the various datasets via an iterative updating scheme based on Newton’s method. For the first three hedging scenarios, the M7-M5 model (3) produces the lowest BIC values. For the Taiwan pension plan hedged by a Japan index, and also the Singapore pension plan hedged by a Hong Kong index, the bivariate Lee-Carter model (1) gives the lowest BIC, followed very closely by the common factor model (2). Although the BIC is probably the most frequently used selection criterion in the mortality projection modelling literature, it is fundamentally a measure on how good the past patterns are being captured under parsimonious use of parameters, which may or may not lead to accurate or reasonable future predictions. Moreover, as noted earlier, there is always some level of uncertainty in model selection, and the BIC alone (and even together with residuals examination) may point to a less appropriate model given the random fluctuations in the data and the lack of information on possible outliers. In fact, a number of other model aspects should also be considered (e.g. Cairns et al. 2009). In terms of assessing longevity basis risk here, the most relevant model aspects would be the coherence property, behaviour of simulated future variability, and simulation method.
Table 2.1 BIC values of fitting three two-population mortality projection models to different datasets

<table>
<thead>
<tr>
<th>Family of Mortality Models</th>
<th>Assured Lives vs E&amp;W</th>
<th>Pensioners vs E&amp;W</th>
<th>NZ vs AUS</th>
<th>TAI vs JAP</th>
<th>SG vs HK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lee-Carter</td>
<td>16,948</td>
<td>16,972</td>
<td>21,577</td>
<td>26,895</td>
<td>4,189</td>
</tr>
<tr>
<td>Li-Lee</td>
<td>17,049</td>
<td>17,067</td>
<td>21,694</td>
<td>26,897</td>
<td>4,207</td>
</tr>
<tr>
<td>CBD</td>
<td>16,312</td>
<td>16,376</td>
<td>21,240</td>
<td>29,125</td>
<td>4,524</td>
</tr>
</tbody>
</table>

Table 2.2 summarises the behaviour of the central estimates and simulated variability of each time series process considered. Table 2.3 provides the details of the selected orders of the fitted time series processes for all the datasets. Figure 2.5 then shows the various time-varying parameters estimated (solid lines), their central estimate projections (dashed lines), and their simulated 95% prediction intervals (dotted lines) using the UK assured lives data under different mortality projection models and time series processes. As shown, the random walk, co-integrated, VECM, and integrated autoregressive processes all produce linear projected (central estimate) trends and increasing variability across time in the simulations, whereas the (weakly stationary) autoregressive processes show convergence and bounded variability. The widths of the prediction intervals vary between different time series processes. For example, within the Lee-Carter family, the VECM(2) generates narrower intervals for both $\kappa_{1,1}$ and $\kappa_{1,2}$, and the VARIMA(1,1,0) produces wider intervals for $\kappa_{1,2}$ but narrower intervals for $\kappa_{1,1}$. For the Li-Lee family, the AR(3) yields wider (bounded) intervals for $\kappa_{1,1,1}$ compared to the AR(1) and VAR(1). For the CBD family, the VARIMA(2,1,0) leads to wider intervals for $\kappa_{1,1}^R$, $\kappa_{1,2}^R$, and $\kappa_{1,3}^R$ compared to the multivariate random walk with drift, and the bivariate random walk without drift leads to unbounded intervals for $\kappa_{1,1}^R$ and $\kappa_{1,2}^R$, in contrast to the bounded intervals from the VAR(2).
Table 2.2  Major characteristics of selected time series processes under different mortality projection models

<table>
<thead>
<tr>
<th>Model Choice</th>
<th>Temporal Parameters</th>
<th>Time Series Process</th>
<th>Central Estimates</th>
<th>Future Variability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1a) non-coherent</td>
<td>$\kappa_{1,1}, \kappa_{1,2}$</td>
<td>BRWD</td>
<td>linear trends</td>
<td>increasing</td>
</tr>
<tr>
<td>(1b) non-coherent</td>
<td>$\kappa_{1,1}, \kappa_{1,2}$</td>
<td>co-integrated</td>
<td>linear trends</td>
<td>increasing</td>
</tr>
<tr>
<td>(1c) non-coherent</td>
<td>$\kappa_{1,1}, \kappa_{1,2}$</td>
<td>VECM($p$)</td>
<td>long-term linear trends</td>
<td>increasing</td>
</tr>
<tr>
<td>(1d) non-coherent</td>
<td>$\kappa_{1,1}, \kappa_{1,2}$</td>
<td>VARIMA($p,1,0$)</td>
<td>long-term linear trends</td>
<td>increasing</td>
</tr>
<tr>
<td>(2)</td>
<td></td>
<td>RWD</td>
<td>linear trend</td>
<td>increasing</td>
</tr>
<tr>
<td>(2a) coherent</td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>AR($1$)</td>
<td>convergence</td>
<td>bounded</td>
</tr>
<tr>
<td>(2b) coherent</td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>AR($p$)</td>
<td>convergence</td>
<td>bounded</td>
</tr>
<tr>
<td>(2c) coherent</td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>correlated AR($p$)</td>
<td>convergence</td>
<td>bounded</td>
</tr>
<tr>
<td>(2d) coherent</td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>VAR($p$)</td>
<td>convergence</td>
<td>bounded</td>
</tr>
<tr>
<td>(3a) coherent</td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>MRWD</td>
<td>linear trends</td>
<td>increasing</td>
</tr>
<tr>
<td></td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>VAR($p$)</td>
<td>convergence</td>
<td>bounded</td>
</tr>
<tr>
<td>(3b) coherent</td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>VARIMA($p,1,0$)</td>
<td>long-term linear trends</td>
<td>increasing</td>
</tr>
<tr>
<td>(3c) coherent</td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>MRWD</td>
<td>linear trends</td>
<td>increasing</td>
</tr>
<tr>
<td>(3d) coherent</td>
<td>$\kappa_{1,1}, \kappa_{2,1}$</td>
<td>correlated MRWD</td>
<td>linear trends</td>
<td>increasing</td>
</tr>
</tbody>
</table>

Note: The terms RWD, BRWD, MRWD, and BRW stand for the random walk with drift, bivariate random walk with drift, multivariate random walk with drift, and bivariate random walk without drift respectively.

Figure 2.6 plots the corresponding book-to-reference ratios of projected death rates at ages 75 and 85. In agreement with the descriptions in the previous section, models (1a) to (1d) produce divergent ratios of projected death rates between the two populations. Moreover, the projected ratios diverge in various directions at different ages even under the same model. By contrast, models (2a) to (2d) and (3a) to (3d) yield convergent ratios of projected death rates at each age, which are in the range of around 0.6 to 0.7 as illustrated in the plots. Figure 2.7 then displays some simulated ratios using 10 randomly picked simulated paths. There are clear differences between model (1a) and models (2a) and (3a). Under model (1a), the simulated ratios can move to values very different from the projected ones over
time, while under models (2a) and (3a), the simulated ratios fluctuate around the projected levels. The potential variations under model (1a) are much greater than those under models (2a) and (3a), in which model (3a) demonstrates more obvious fluctuations but within a shorter distance from the projected values than model (2a). Since demographic basis risk arises from possible deviations between the book and reference mortality movements (due to demographic or socioeconomic differences), the behaviour of these simulated ratios of death rates would have a significant implication on the calculated levels of hedge effectiveness, which will be discussed in the following numerical analysis.

Table 2.3  
Selected orders of fitted time series processes under different mortality projection models for different datasets

<table>
<thead>
<tr>
<th>Model Choice</th>
<th>Temporal Parameters</th>
<th>Assured Lives vs E&amp;W</th>
<th>Pensioners vs E&amp;W</th>
<th>NZ vs AUS</th>
<th>TAI vs JAP</th>
<th>SG vs HK</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1c)</td>
<td>$\kappa_{1,1}, \kappa_{2,2}$</td>
<td>VECM(2)</td>
<td>VECM(2)</td>
<td>VECM(2)</td>
<td>VECM(2)</td>
<td>VECM(2)</td>
</tr>
<tr>
<td>(1d)</td>
<td>$\kappa_{1,1}, \kappa_{2,2}$</td>
<td>VARIMA(1,1,0)</td>
<td>VARIMA(1,1,0)</td>
<td>VARIMA(1,1,0)</td>
<td>VARIMA(2,1,0)</td>
<td>VARIMA(2,1,0)</td>
</tr>
<tr>
<td>(2b)</td>
<td>$\kappa_{1,2}, \kappa_{2,2}$</td>
<td>AR(3), AR(1)</td>
<td>AR(2), AR(3)</td>
<td>AR(2), AR(1)</td>
<td>AR(3), AR(1) / AR(2), AR(1)</td>
<td>AR(1), AR(1)</td>
</tr>
<tr>
<td>(2c)</td>
<td>$\kappa_{1,2}, \kappa_{2,2}$</td>
<td>AR(3), AR(1)</td>
<td>AR(2), AR(3)</td>
<td>AR(2), AR(1)</td>
<td>AR(3), AR(1) / AR(2), AR(1)</td>
<td>AR(1), AR(1)</td>
</tr>
<tr>
<td>(2d)</td>
<td>$\kappa_{1,2}, \kappa_{2,2}$</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
<td>VAR(2)</td>
<td>VAR(1) / VAR(1)</td>
<td>VAR(1)</td>
</tr>
<tr>
<td>(3a)</td>
<td>$\kappa_{1,1}, \kappa_{2,2}$</td>
<td>VAR(2)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
</tr>
<tr>
<td>(3b)</td>
<td>$\kappa_{1,1}, \kappa_{2,2}$</td>
<td>VARIMA(2,1,0)</td>
<td>VARIMA(2,1,0)</td>
<td>VARIMA(1,1,0)</td>
<td>VARIMA(1,1,0)</td>
<td>VARIMA(1,1,0)</td>
</tr>
<tr>
<td>(3b)</td>
<td>$\kappa_{1,1}, \kappa_{2,2}$</td>
<td>VAR(2)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
</tr>
<tr>
<td>(3d)</td>
<td>$\kappa_{1,1}, \kappa_{2,2}$</td>
<td>VAR(2)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
<td>VAR(1)</td>
</tr>
</tbody>
</table>

Note: The selected orders of the fitted time series processes are based on the partial autocorrelation functions and matrices, and whether the autocorrelations and cross-correlations of the residuals are insignificant, the estimated parameters are significant, and the resulting fitted time series process (2b, 2c, 2d, 3a, 3b, 3d) is weakly stationary. For the Taiwan pension plan hedged by a Japan index, there are two additional factors in the common factor model, while for all the other hedging scenarios, there is only one additional factor. The number of additional factors is based on the lowest BIC value.
Figure 2.5  Time series projections and 95% prediction intervals from (bivariate) residuals bootstrapping using assured lives (book) data and England and Wales (reference) data
Figure 2.5 Continued
Figure 2.6  Projected (and observed) book-to-reference ratios of central death rates at ages 75 and 85 using assured lives (book) data and England and Wales (reference) data
Figure 2.7  Simulated (and observed) book-to-reference ratios of central death rates at ages 75 and 85 from semi-parametric bootstrapping using assured lives (book) data and England and Wales (reference) data

Figure 2.8 offers a different view on the simulated book and reference death rates at ages 80 and 85 at different points of time using the UK pensioners data. It can be seen that over time, the simulated death rates move to the southwest direction due to mortality improvements, and the simulated variability (both the vertical and horizontal ranges) increases because of higher uncertainty in more distant future. But again, there are significant differences between the three models. For models (2a) and (3a), the dependence between the simulated book and reference death rates increases gradually with time, whereas for model
(1a), the weak dependence appears to remain at a low level. It means that for the former the two death rates are very unlikely to deviate significantly further from each other in the long term, but for the latter the two death rates can move in varying directions. The different model structures clearly have a large impact on the resulting association between the two populations in the simulations. The discussion of the numerical study below will explore the underlying reasons for these observations.

Table 2.4 sets out the estimates of the standard deviation and the 99.5% Value-at-Risk (VaR) (minus the mean) of the present value of the pension plan liability, as a percentage of the expected present value of the liability. On average, the standard deviation is about 1.7% and the 99.5% VaR is around 4.3% of the mean, in which the ratio between the two measures is close to that of a standard normal distribution. For the assured lives and pensioners datasets, the bootstrapping approaches generally give larger VaR estimates than those from Monte Carlo simulation, while for the other datasets, the differences are less obvious. The smaller sizes of the former may be the main reason why they demonstrate a greater effect of parameter uncertainty. The simulated variability from the Li-Lee family (2a, 2b, 2c, 2d), but not the other two families, is fairly robust to the selection of time series process. Model (3c) produces by far the largest estimates among all the models. It appears that the bivariate random walk for $K_{t}^{B}$ together with the multivariate random walk for $K_{t}^{R}$ in model (3c) produces much more variability in the book simulations compared with the other models.

In line with Coughlan et al. (2011) and Li et al. (2017), the level of hedge effectiveness is defined as $[1 - \text{risk}_{\text{hedged}} / \text{risk}_{\text{unhedged}}] \times 100\%$. The two quantities $\text{risk}_{\text{unhedged}}$ and $\text{risk}_{\text{hedged}}$ are the pension plan’s longevity risk exposure before and after implementing the hedge. This measure gives the proportion of the original longevity risk exposure that is being transferred away. The remaining risk can then be seen as a result of longevity basis risk. We consider the longevity risk exposure as the standard deviation and also the 99.5% VaR minus the mean of the present value of the pension plan liability. Note that the 99.5% VaR measure is highly relevant to the Solvency Capital Requirement (SCR) calculation under Solvency II. Figure 2.9 presents the levels of hedge effectiveness (i.e. the proportion of the initial risk that is reduced) under various models and simulation methods using different datasets. The major observations and implications from the numerical results are listed below:
Figure 2.8  Simulated book and reference central death rates at ages 80 and 85 in 5, 10, 15, and 20 years from residuals bootstrapping using pensioners (book) data and England and Wales (reference) data

Model (1a)

Model (2a)
Table 2.4  Estimates of standard deviation and 99.5% VaR (minus mean) of pension plan liability (in % of mean)

<table>
<thead>
<tr>
<th>Model Choice</th>
<th>Assured Lives</th>
<th>Pensioners</th>
<th>NZ</th>
<th>TAI</th>
<th>SG</th>
</tr>
</thead>
<tbody>
<tr>
<td>SD</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1a)/(1b)/(1c)/(1d)</td>
<td>1.3/1.1/0.9/1.4</td>
<td>2.3/1.5/1.1/1.6</td>
<td>1.8/1.5/1.1/1.3</td>
<td>1.9/1.7/1.1/0.9</td>
<td>1.7/2.0/1.1/1.1</td>
</tr>
<tr>
<td>(2a)/(2b)/(2c)/(2d)</td>
<td>1.1/1.1/1.1/1.1</td>
<td>1.4/1.3/1.4/1.4</td>
<td>1.4/1.4/1.4/1.4</td>
<td>1.7/1.7/1.7/1.7</td>
<td>1.6/1.6/1.7/1.6</td>
</tr>
<tr>
<td>(3a)/(3b)/(3c)/(3d)</td>
<td>1.6/1.6/2.5/1.6</td>
<td>1.9/1.8/3.1/1.7</td>
<td>2.0/1.4/3.2/2.0</td>
<td>1.9/1.4/2.6/1.8</td>
<td>2.5/1.7/3.4/2.2</td>
</tr>
<tr>
<td>VaR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1a)/(1b)/(1c)/(1d)</td>
<td>3.1/1.1/0.9/1.4</td>
<td>5.6/3.8/2.7/3.9</td>
<td>4.5/3.6/2.8/3.4</td>
<td>4.6/4.4/2.8/2.3</td>
<td>4.2/4.8/2.7/2.7</td>
</tr>
<tr>
<td>(2a)/(2b)/(2c)/(2d)</td>
<td>2.6/2.7/2.7/2.7</td>
<td>3.5/3.3/3.3/3.4</td>
<td>3.5/3.5/3.4/3.4</td>
<td>4.3/4.3/3.4/3.4</td>
<td>3.9/3.9/4.1/3.9</td>
</tr>
<tr>
<td>(3a)/(3b)/(3c)/(3d)</td>
<td>3.8/3.9/5.9/3.8</td>
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<td>5.1/3.5/7.5/5.0</td>
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1. The estimated levels of hedge effectiveness are largely between 50% to 80%. The clear exceptions include those produced from models (1a), (1d), and (3c), which are about 20% or lower. The first two of these three cases are non-coherent and generate increasing variability in the time-varying parameters’ future simulations for both the book and reference populations. Although the estimated correlations in their Gaussian error terms are quite high (about 0.3 or greater), the relationship between the book and reference mortality levels under the two models looks too weak, when considering their past movements in the data (see Figure 2.3), and there is an overestimation of longevity basis risk. Comparatively, while models (1b) and (1c) are also non-coherent, their co-integration and error correction structures lead to some extent of co-movements between the two populations in the simulations, which is reflected in the estimated hedge effectiveness. The last exception, model (3c), is coherent, but it produces increasing simulated variability over time for the ‘book minus reference’ component’s time-varying parameters. The resulting effect is that the simulated book and reference mortality movements could deviate significantly from each other, progressively so over the long run, leading to an overestimation of longevity basis risk and so an underestimation of hedge effectiveness. By contrast, the other model choices do not suffer from this problem.

2. The results from the Li-Lee family (2a, 2b, 2c, 2d) are generally quite robust to the choice of time series modelling, as long as the fitted time series processes are weakly stationary. Under the Li-Lee mortality structure, both populations are governed by the common factor, in which the common mortality index has increasing variability over time in the simulations, while the additional population-specific factors’ time-varying parameters have bounded variability in the simulations under all the four choices. Consequently, the former variability would dominate eventually, and any deviance between the simulated book and reference mortality movements is unlikely to continue to grow. The figures from models (2b) and (2c) are slightly lower than those from model (2a) (by only about 3% in magnitude on average; more obvious for the pensioners dataset), due to the higher selected orders in the former. But in general there is not much difference in the results between models (2b) and (2c), and also between models (2a) and (2d), which suggest that the additional dependence (via
correlation or vector autoregression) is unlikely to have a material impact on assessing longevity basis risk. Besides, though the common factor provides a convenient way to capture the link between the two populations, the common factor model (2) requires using the same data length for both populations in maximum likelihood estimation. In practice, the length of book data is usually shorter, and approximation methods or Bayesian techniques may be adopted to fit the model under the presence of missing data.

3. The figures from models (3a), (3b), and (3d) of the CBD family tend to be higher than those from models (2a) and (2d) (by about 10% in magnitude or more) for the assured lives and New Zealand portfolios, while the two sets of results differ more randomly for the other datasets. Under these models, the reference component’s temporal parameters have increasing simulated variability over time, while the ‘book minus reference’ component’s temporal parameters have bounded simulated variability instead. The variability in the reference component would then dominate in the long term, whereas the variability in the second component would become less influential comparatively. The consequence is that the book and reference mortality movements cannot deviate indefinitely in the simulated scenarios. The results are about the same between models (3a) and (3d), which again suggest that the additional dependence (via correlation) does not have an obvious effect on measuring longevity basis risk.

For model (3b), as noted previously, the prediction intervals for $\kappa_{t,1}^g$, $\kappa_{t,2}^g$, and $\kappa_{t,3}^g$ based on the VARIMA process are often different to those based on the multivariate random walk with drift in model (3a). Some subsequent effect can be seen in the differences between the figures from models (3a) and (3b), in which the latter ones tend to be lower.

4. The hedging results estimated from Monte Carlo simulation tend to be higher than those from the bootstrapping approaches (by about 7% in magnitude on average) for the assured lives and pensioners datasets. Since performing Monte Carlo simulation directly on the error terms of the time series processes allows for only process error but not parameter error, there would be an underestimation of longevity basis risk and hence an overestimation of hedge effectiveness. But for the other datasets (New
Zealand, Taiwan, and Singapore), there are no such obvious differences, which indicate that the effect of parameter uncertainty is not material for these proxy book data with a large size and more stable population compositions and mortality patterns. In addition, there are no clear, significant differences between the results estimated from the two residuals bootstrapping approaches, in which one involves resampling the reference and book residuals separately and the other groups together the two sets of residuals in each age-time cell as an individual data point for resampling. It means that the additional dependence from linking the residuals does not appear to have any effect on longevity basis risk calculation. Some random differences in the results between the semi-parametric bootstrapping and the residuals bootstrapping can also be seen, but most of them are small and do not show any particular patterns.

5. For a demonstration of applying the modified semi-parametric bootstrapping approach, we integrate models (1b), (2a), and (3a) which have earlier been shown to produce more reasonable estimates of hedge effectiveness. Following Yang et al. (2015), the optimal model for each pseudo data sample in the bootstrapping process is selected on the basis of the BIC. That is, the BIC values of fitting the three mortality projection models are compared, and the one with the lowest value is chosen for that pseudo data. Note that besides a single statistical criterion, a mix of other quantitative and even qualitative criteria may also be used, though the selection rules will then be more complex and the computation time will lengthen. Table 2.5 gives the proportions of different models being chosen out of 5,000 scenarios in each case. For the assured lives, pensioners, and New Zealand datasets, model (3a) dominates in all the simulated scenarios. For the Taiwan portfolio, model (2a) is selected in 99% of the scenarios and model (1b) is selected in only 1%. For the Singapore portfolio, models (2a) and (1b) share a split of 58% and 42%. Furthermore, as shown in Figure 2.9 (last row), regarding the Singapore portfolio, the final hedging results from this approach may be perceived as a ‘weighted’ average of those calculated separately from models (1b) and (2a), in which the ‘weights’ are determined by how well the two models are fitted to each of the 5,000 pseudo data samples.
6. The estimated levels of hedge effectiveness are quite close between using the standard deviation and 99.5% VaR in most cases. This observation may result from the fact that the simulated distributions of the pension plan liability are fairly symmetric and do not have a heavy tail, under all the mortality projection models, time series processes, and simulation methods considered, both before and after hedging. We have also checked the results based on the 95% VaR and the observations are similar. Note that if certain additional features such as structural mortality changes and mortality jumps are incorporated into the models, the resulting distributions may be more skewed or have heavier tails, which would probably make the calculated levels of hedge effectiveness more varying between different percentiles.

7. When the size of the pension plan is reduced to 1,000 members, the levels of hedge effectiveness estimated from those coherent models drop to mostly around 20% to 40% (not shown here). By contrast, when the pension plan size is infinite (i.e. the step of using the binomial distribution to simulate the number of lives is omitted), the estimated levels of hedge effectiveness largely rise by about 10% or more in magnitude from the initial setting of 10,000 members. The effect of sampling basis risk is significant, which pinpoints that index-based longevity hedging would be more feasible for either very large pension plans, foundations joined by small pension plans, or reinsurers who have accumulated sizable longevity risk exposures from smaller insurers and pension plans.

8. There are some differences in the hedging results between using the five datasets. On average, after integrating both process error and parameter error via the bootstrapping approaches, the highest level of hedge effectiveness is given by using the New Zealand dataset, followed by the Taiwan, assured lives, Singapore, and pensioners datasets. It has been mentioned earlier that parameter error would be immaterial for book data with stable population compositions and mortality patterns, which may explain some of the results here. Another possible reason for the greater hedge effectiveness shown by the New Zealand pension plan hedged by an Australia index is the geographical and cultural closeness between Australia and New Zealand. Both neighbours are island nations in the South Pacific and have very close connections...
historically, socially, and economically, and they would tend to have more concurrent mortality movements than otherwise.

9. Under models (2a), (2b), (2c), (2d), (3a), (3b), and (3d), the calculated levels of hedge effectiveness of the annual cash flows (not the present value; not shown here) are indeed very low in the early years, but then increase significantly across time. That is, the dependence between the two populations’ mortality levels grows with time in the simulations. For the Li-Lee model types, the common mortality index has increasing variability, while the additional time-varying parameters have bounded variability. Similarly, for the M7-M5 model types, the reference component’s time-varying parameters have increasing variability, but the book component’s time-varying parameters have bounded variability. The resulting model effect is that the simulated variability of the differences between both populations would have lesser impact gradually. Hence the association between the two populations increases over time (see Figure 2.8).

10. The weights of the S-forwards are numerically optimised to maximise the hedge effectiveness. Excluding models (1a), (1d), and (3c) which generate unreasonable simulations, for the assured lives dataset, the weights are roughly about 0.5 (per person) in the first half of the age range, 0.7 in the third quarter, and 0.9 in the last quarter. For the pensioners dataset, the weights are around 0.7 in the first half of the age range and 0.9 in the second half. For the New Zealand, Taiwan, and Singapore datasets, most of the weights are approximately equal to one for the whole range. Interestingly, the major patterns in the weights estimated are not too different between the various models. It should be noted that these weights are subject to the technical limitations of the optimisation process, uncertainty of the model choice, and random variations in the simulated samples. And in practice, it would be impossible to enter into such numerically precise amounts of S-forward positions due to liquidity issues. Accordingly, we have also tested the hedging results using the approximate overall weights noted above, and we realise that the corresponding reductions in the levels of hedge effectiveness are actually quite small, mostly being a few percent in magnitude.
Figure 2.9  Estimated levels of hedge effectiveness
Figure 2.9  Continued
Figure 2.9  Continued
Table 2.5  Simulated proportions of model selections in modified semi-parametric bootstrapping for different datasets

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2.4 Further remarks

In this Section 2, we study the impact of model uncertainty on hedging longevity risk via index-based derivatives and measuring longevity basis risk. We adopt three families of mortality projection models, using a number of time series processes and simulation methods, to calibrate index-based S-forwards, construct longevity hedges, and assess the resulting hedge effectiveness. Overall, the estimated levels of hedge effectiveness are mostly about 50% to 80% for a large pension plan, which broadly agree with the figures estimated in Li et al. (2017) using different models and datasets to those adopted here, but the precise levels still depend heavily on the selected model. Apart from comparing the results from different models, the semi-parametric bootstrapping approach can also be extended to include model selection criteria and test several models simultaneously. However, those models or time series processes with unrestricted simulated variability in the differences between the book and reference populations lead to irrational hedging results and should be avoided. In practice, the final model choice relies not only on a quantitative analysis of (usually limited) past book data but also on some extent of qualitative judgement about uncertain future mortality trends and the user’s own knowledge or preference.

There are a few possible areas for future research. So far, we focus on static hedging, under which the hedging tools are not rebalanced in the future. By contrast, dynamic hedging requires constant rebalancing of the hedged portfolio, and it would be interesting to explore the feasibility of dynamic index-based longevity hedging. For simplicity, we also assume that the forward rates are equal to the central estimates, implying a zero premium on longevity risk. In order to further incorporate the costs of longevity hedging, it would be useful to investigate different risk-neutral methods (e.g. Cairns et al. 2006b) for pricing index-based derivatives. Moreover, Li et al. (2017) considered some simple model extensions to take structural mortality changes and mortality jumps into account when assessing the hedge effectiveness. More sophisticated methods such as regime-switching models (e.g. Milidonis et al. 2011) and copulas (e.g. Wang et al. 2015) can further be tested in modelling these extreme events. In addition, we use a single information criterion in the modified semi-parametric bootstrapping process. Alternatively, a more complex set of rules consisting of multiple quantitative and qualitative criteria may provide a more realistic view of model uncertainty. Lastly, Bayesian techniques can also be adopted to allow for model uncertainty.
directly, instead of using the modified bootstrapping approach above. For instance, two or more competing models can be integrated into the Bayesian framework by setting a prior distribution for all the model candidates (e.g. Cairns 2000).
3 Parametric Mortality Indices and Out-of-the-Money Longevity Hedges

Proposed by Chan et al. (2014), parametric mortality indices (i.e. indices created from the time-varying parameters of a suitable stochastic mortality projection model) can be used to develop tradable mortality-linked derivatives such as K-forwards. Compared to existing indices such as the LLMA’s LifeMetrics, parametric mortality indices are richer in information content, allowing the market to concentrate liquidity more readily. In this section, we study this concept further in several aspects. First, we consider options written on parametric mortality indices. Such options enable hedgers to create out-of-the-money longevity hedges, which, compared to at-the-money-hedges using \( q \)-forwards or K-forwards, may better meet hedgers’ need for protection against downside risk. Second, using the properties of the time-series processes for the parametric mortality indices, we derive analytical risk-neutral pricing formulas for K-forwards and options. These formulas remove the need for computationally intensive nested simulations that are entailed in, for example, the calculation of the hedging instruments’ values when a dynamic hedge is adjusted. Finally, we construct static and dynamic Greek hedging strategies using K-forwards and options, and demonstrate empirically the conditions under which an out-of-the-money hedge is more economically justifiable than an at-the-money hedge.

The theoretical background of parametric mortality indices and K-forwards are given in Section 3.1. The CBD mortality indices and their merits are reviewed in Section 3.2. The risk-neutral process for the CBD mortality indices are defined in Section 3.3. The derivation of the exact analytical risk-neutral pricing formulas for K-forwards and options are presented in Section 3.4. The liability being hedged and the approximation method for valuing it are specified in Section 3.5. Our theoretical work on static and dynamic delta hedging using K-forwards and options are set forth in Section 3.6. Numerical illustrations and sensitivity tests are provided in Section 3.7. Conclusions and suggestions for future research are set out in Section 3.8. More derivation details of the formulae are given in Section 3.9.
3.1 Parametric mortality indices

The notion of parametric mortality indices was originally proposed by Chan et al. (2014). Parametric mortality indices are created from the time-varying parameters of an appropriately chosen stochastic mortality projection model. They summarise how the mortality curve of a certain population evolves over time, and the random deviations from the expected trajectories reflect the level of longevity risk that the population is subject to. Chan et al. (2014) further proposed a conceptual security called K-forward, which refers to a zero-coupon swap that is written on a specific parametric mortality index. Through K-forwards, pension plan providers can hedge their longevity risk exposures, while capital market investors can tap into the longevity asset class which offers them risk premium and diversification benefits.

Compared to non-parametric mortality indices such as the LLMA’s LifeMetrics and Deutsche Börse’s Xpect Cohort Indexes, parametric mortality indices are advantageous of being richer in information content, so that non-parallel shifts in the underlying mortality curve over time can be captured by only a few indices. This property can enable the market to better concentrate liquidity, fostering the popularity of less costly index-based longevity hedging solutions. Further, depending on the model from which they are created, parametric mortality indices are often highly interpretable. For instance, the two parametric mortality indices (the CBD mortality indices) recommended by Chan et al. (2014) can be understood as the level and gradient of the (logit-transformed) underlying mortality curve, respectively. The ease of interpretation makes K-forwards and other securities written on parametric mortality indices more marketable to capital market investors.

Since proposed by Chan et al. (2014), parametric mortality indices and K-forwards have been further studied by various researchers. Tan et al. (2014) examined how a static K-forward hedge may be calibrated with a ‘duration-matching’ approach, and demonstrated that a K-forward hedge has a potential to outperform a q-forward hedge in terms of the number of hedging instruments required. Hau et al. (2017) studied the counterparty credit risk associated with K-forwards. In particular, they derived the risk-neutral default probability (from the hedge provider’s perspective) and calculated the credit value adjustment in different circumstances. Wei (2017) modelled the CBD mortality indices with the LLCBD model proposed by Liu and Li (2017) and performed a further study on the counterparty credit risk.
concerning K-forwards. Biffis et al. (2017) introduced a security that is written on the population-specific time-varying parameters of the augmented common factor model (Li and Lee 2015). Although they name their proposed security slightly differently (‘k-forward’ instead of ‘K-forward’), the spirit behind (i.e. writing securities on the time-varying parameters of a stochastic mortality projection model) is essentially the same.

In the following sections, we revisit the concepts of parametric mortality indices and K-forwards, with a focus on three specific objectives. First, we explore security structures other than a zero coupon swap and examine how the alternative security structures may benefit the hedger. Second, we study the risk-neutral valuation of K-forwards and other securities written on parametric mortality indices. This objective distinguishes our work from the previous ones on parametric mortality indices by Biffis et al. (2017) and Tan et al. (2014), who ignored the cost of hedging completely. Third, we consider dynamic hedging with K-forwards and options, extending the work in Tan et al. (2014) who focused on static hedging and used forward contracts only. We choose to focus on the CBD mortality indices, which satisfy the three key criteria set out by Chan et al. (2014).

In terms of security structures, we particularly focus on options written on parametric mortality indices (thereafter referred to as K-options). Longevity securities with option-like payoffs were discussed in the early work of Blake et al. (2006) and Cairns et al. (2006a, 2008) and have received considerable attention recently (e.g. Cairns and Boukfaoui 2017). As Michaelson and Mulholland (2014) pointed out, longevity securities featuring option-like payoffs enable hedgers to build out-of-the-money hedges against extreme downside longevity outcomes, while retaining the potential upside gain. We investigate the feasibility of building out-of-the-money longevity hedges using K-options and examine how such hedges perform compared to their at-the-money counterparts in different circumstances. A similar study concerning payoff structures of longevity securities was conducted by Biffis and Blake (2010), who concluded that in some cases it is optimal to transfer longevity risk using a call (put) option on mortality (survival) rates. It should be noted that their conclusion is based on the assumption of a Walrasian economy, whereas in our investigation we assume that hedgers (buyers of K-options) are price takers. Another noteworthy distinction is that the liability being hedged in our set-up is apparently more realistic than the ‘annuity-like’ liability that Biffis and Blake (2010) considered.

In terms of pricing, we strive to derive analytical (risk-neutral) pricing formulas for
K-forwards and K-options. When considering derivatives on mortality and survival rates (e.g. $q$-forwards and $S$-forwards), analytical pricing formulas are available only under rather restrictive setups (e.g. Dahl 2004, Bauer et al. 2010, 2013). In more general situations, prices of derivatives on mortality and survival rates have to be estimated using simulations (e.g. Boyer and Stentoft 2013, Li 2010, Li and Ng 2011) or closed-form expressions that involve a certain extent of approximations (e.g. Wang and Yang 2013). Nevertheless, thanks to the fact that K-forwards and K-options are written directly on the time-varying parameters which are assumed to follow some tractable time-series processes, analytical pricing formulas for these securities can be derived readily using the structural and statistical properties of the risk-adjusted version of the assumed processes. On the basis of the risk-neutral Cairns-Blake-Dowd model specified by Cairns et al. (2006b), we derive analytical pricing formulas for K-forwards and K-options written on the CBD mortality indices. The resulting formulas are intuitive and satisfy the put-call parity relationship as the Black-Scholes formula for equity options does.

In terms of hedging, we consider both static and dynamic hedging using the CBD mortality indices. Our work on dynamic hedging draws from the previous contributions of Cairns (2011) and Zhou and Li (2017a, 2017b), who avoided the need for the computationally demanding nested simulations entailed at each time point when the hedge is rebalanced using the ‘approximation of survival function’ method. One drawback of this method, as Cairns (2011) mentioned, is that it does not work satisfactorily when the hedging instruments have a non-linear payoff structure. However, this drawback does not apply to our K-options, as their prices and Greek letters can be calculated analytically (without any approximation). We consider both cash flow hedges (which focus on the variability of the cash flows arising from the hedged position) and value hedges (which focus on the variability of the values of hedged position at a certain future time point). For both, we measure the hedge effectiveness in terms of the value-at-risk (VaR), which takes the cost of hedging into consideration.

We provide real data illustrations of the synthesis of our theoretical contributions. Through a range of sensitivity tests, we arrive at several interesting conclusions. For instance, a K-put hedge is more likely to yield a lower VaR than a K-forward hedge when the market prices of risk are high and / or the times-to-maturity of the hedging instruments are long.

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2 For instance, the assumed model in the setup of Bauer et al. (2010) may yield negative forces of mortality.
Another example is that a value hedge is highly robust to changes in the market prices of risk, the risk-free interest rate, and the times-to-maturity of the hedging instruments.
3.2 CBD mortality indices

The CBD mortality indices are developed from the original version of the Cairns-Blake-Dowd model (Cairns et al. 2006b), which can be expressed as

\[
\ln \left( \frac{q_{x,t}}{1 - q_{x,t}} \right) = \kappa_t^{(1)} + \kappa_t^{(2)} (x - \bar{x})
\]

(1)

where \(q_{x,t}\) is the probability that an individual aged \(x\) at time \(t\) will die between \(t\) and \(t + 1\), \(\bar{x}\) is the mean age over the sample age range, and \(\kappa_t^{(1)}\) and \(\kappa_t^{(2)}\) are the time-varying parameters. The time-\(t\) values of the first and second CBD mortality indices are defined as \(\kappa_t^{(1)}\) and \(\kappa_t^{(2)}\) respectively. The first index (\(\kappa_t^{(1)}\)) represents the level of the logit-transformed mortality curve (the curve of \(q_{x,t}\) in year \(t\))\(^3\). A reduction in \(\kappa_t^{(1)}\) implies an overall reduction in mortality, affecting the death probabilities (in logit scale) at all ages by the same amount. The second index (\(\kappa_t^{(2)}\)) represents the gradient of the logit-transformed mortality curve. A change in \(\kappa_t^{(2)}\) reflects a change in the distribution of mortality improvements across age. In particular, an increase in \(\kappa_t^{(2)}\) means that mortality (in logit scale) at younger ages (below the mean age \(\bar{x}\)) improves more rapidly than at older ages (above the mean age \(\bar{x}\)). Chan et al. (2014) further defined \(K_1\) and \(K_2\) risks as the risks surrounding the future values of \(\kappa_t^{(1)}\) and \(\kappa_t^{(2)}\) respectively. They noted that pension plans and life insurers are subject to very different profiles of \(K_1\) and \(K_2\) risks.

For closed pension plans, financial obligations are positively related to future mortality improvements at old ages. Therefore, the worst situation for them occurs when (i) the overall mortality improvement is faster than expected (which happens when the first CBD index is lower than expected) and (ii) the mortality improvement is more concentrated at older ages (which happens when the second CBD index is lower than expected). It follows that the plans are subject to a downside \(K_1\) risk and a downside \(K_2\) risk.

For insurance companies with a concentration on term-life business, the customers of which are typically young, financial obligations are negatively associated with future mortality improvements at young ages. Hence, the worst outcome for them occurs when (i) the overall mortality improvement is slower than expected (which happens when the first CBD index is higher than expected) and (ii) the mortality improvement is more concentrated at older ages (which happens when the second CBD index is lower than expected). These companies are subject to an upside \(K_1\) risk and a downside \(K_2\) risk.

\(^3\) The logit transformation of a real number \(w\) is given by \(\ln(w/(1-w))\).
The above descriptions are summarised in Figure 3.1. This diagram aids us in determining the appropriate positions of the various hedging instruments that are introduced in Section 3.4.

**Figure 3.1**  A graphical illustration of K1 and K2 risks

As Chan et al. (2014) mentioned, the CBD mortality indices possess the following three desirable properties:

1. Small in dimension, but able to represent the age-pattern of mortality improvements

As demonstrated in Figure 3.2, the two CBD mortality indices can represent different patterns of the underlying mortality curve. To achieve this ability with non-parametric indices such as the LLMA’s LifeMetrics, a much larger number of indices would be needed. This property enables effective hedging of longevity risk (which arises from random non-parallel shifts in the underlying mortality curve), while allowing the market to concentrate liquidity on a small number of indices.
2. **Interpretable**
As discussed earlier, the two CBD mortality indices are readily interpretable. The impact of changes in each CBD mortality index on closed pension plans and life insurers can also be clearly understood. This property facilitates marketability of the indices to both hedgers and capital market investors.

3. **The new-data-invariant property**
The new-data-invariant property refers to the condition that when an additional year of data becomes available and the model from which the parametric mortality indices are derived is updated accordingly, the indices in previous years would not be affected. This property is crucial because it would be impossible to track an index if its historical values are revised from time to time. Due to its ‘separable’ likelihood function and the fact that it is free of any identifiability constraints, the original Cairns-Blake-Dowd model (from which the CBD mortality indices are derived) satisfies the new-data-invariant property (see Chan et al. 2014). Note that among the six stochastic mortality projection models considered by Dowd et al. (2010), the original Cairns-Blake-Dowd model is the only one that satisfies the new-data-invariant property.

**Figure 3.2** Curves of $q_{x,t}$ for different pairs of CBD mortality indices at time $t$
3.3 Risk-neutral CBD model

Under both the real-world and risk-neutral probability measures, the Cairns-Blake-Dowd model links $q_{x,t}$ to the two time-varying parameters $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ in exactly the same manner (through equation (1)). However, the stochastic processes driving the dynamics of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ under the two probability measures are different. Following Cairns et al. (2006b), we assume that under the real-world probability measure, $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ follow a bivariate random walk with a constant drift vector:

$$\kappa_t = \mathbf{\mu} + \kappa_{t-1} + \mathbf{A} z_t$$

(2)

where $\kappa_t = (\kappa_t^{(1)}, \kappa_t^{(2)})'$ represents the vector of the time-varying parameters, $\mathbf{\mu} = (\mu_t^{(1)}, \mu_t^{(2)})'$ is the constant drift vector, $\mathbf{A}$ is a 2-by-2 upper-triangular matrix, and $z_t = (z_t^{(1)}, z_t^{(2)})'$ is a vector of two uncorrelated standard normal random variables under the real-world probability measure. It is further assumed that $z_s$ and $z_u$ are uncorrelated if $s \neq u$.

Following Cairns et al. (2006b), we further assume that the risk-neutral dynamics of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ are driven by

$$\kappa_t = \mathbf{\mu} + \kappa_{t-1} + \mathbf{A} (\mathbf{\tilde{z}}_t - \lambda),$$

where $\mathbf{\tilde{z}}_t = (\mathbf{\tilde{z}}_t^{(1)}, \mathbf{\tilde{z}}_t^{(2)})'$ is a vector of two uncorrelated standard normal random variables (both of which have no serial correlation) under the risk-neutral probability measure, $\lambda = (\lambda_t^{(1)}, \lambda_t^{(2)})'$ represents the vector of market prices of risk. We can understand $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ as the market prices of K1 and K2 risks respectively. Rearranging this equation, we get

$$\kappa_t = \mathbf{\check{\mu}} + \kappa_{t-1} + \mathbf{A} \mathbf{\check{z}}_t$$

(3)

where $\mathbf{\check{\mu}} = \mathbf{\mu} - \mathbf{A} \lambda$, which says that under the risk-neutral probability measure, $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ also follow a random walk but with a constant drift vector that is different from that under the real-world probability measure. It is interesting to note that the relationship between the real-world process (equation (2)) and the risk-neutral process (equation (3)) we consider is consistent with that obtained using the Esscher transform (see Hunt and Blake 2015).

The market prices of risk are calibrated to market information, for example, market prices of individual life annuities. The calibration work is beyond the scope of this report. In our baseline calculations, we set $\lambda_t^{(1)} = \lambda_t^{(2)} = 0.175$. This collection of market prices of risk was obtained by Cairns et al. (2006b) using the market price of the longevity bond jointly announced by BNP Paribas and the European Investment Bank in 2004. Interested readers

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4 We fully acknowledge that the calibrated market prices of risk are subject to some limitations: (1) the longevity bond was not actually traded, so that its announced price may not truly reflect the market participants’ aversion to longevity risk; (2) the bond was announced more than 10 years ago, so the information contained in
are referred to Hunt and Blake (2015) for a detailed discussion on how the calibration work may be performed in practice.

Finally, it follows from equation (3) that given information up to and including time \( t \), \( \kappa^{(i)}_t \) for \( T > t \) under the risk-neutral probability measure can be expressed as

\[
\kappa^{(i)}_T = \kappa^{(i)}_t + (T - t)\tilde{\mu}^{(i)} + \sum_{k=1}^{T-t} \tilde{v}^{(i)}_{t+k}, \quad i = 1, 2
\]

(4)

where \( \tilde{v}^{(1)}_{t+k} = a_{1,1}z^{(1)}_{t+k} + a_{1,2}z^{(2)}_{t+k} \) and \( \tilde{v}^{(2)}_{t+k} = a_{2,1}z^{(1)}_{t+k} + a_{2,2}z^{(2)}_{t+k} \) with \( a_{ij} \) being the \((i, j)\)th element in \( A \).

Consequently, given information up to and including time \( t \), \( \kappa^{(i)}_t \) for \( T > t \) under the risk-neutral probability measure is normally distributed with a mean of

\[
E^Q_t[\kappa^{(i)}_T] = \kappa^{(i)}_t + (T - t)\tilde{\mu}^{(i)}, \quad i = 1, 2
\]

(5)

and a variance of

\[
\text{Var}^Q_t[\kappa^{(i)}_T] = \begin{cases} 
(T - t)(a_{1,1}^2 + a_{1,2}^2), & i = 1 \\
(T - t)a_{2,2}^2, & i = 2
\end{cases}
\]

(6)

where \( E^Q[\cdot] \) and \( \text{Var}^Q[\cdot] \) represent the risk-neutral expectation and variance given information up to and including time \( t \) respectively. The pricing formulas presented in the next section draw heavily from equations (5) and (6).

its announced price may not be up-to-date. To mitigate these limitations, we sensitivity test (in Section 3.7) the hedging results using a wide range of market prices of risk, including the extreme case when \( \lambda^{(1)} = \lambda^{(2)} = 0 \).
3.4 Hedging instruments

In this section, we define three securities (K-forward, K-call, and K-put) that are written on the CBD mortality indices and present the exact analytical pricing formulas for the three securities on the basis of the previously described risk-neutral process. In the following, we refer to a K-forward written on the first (second) CBD mortality index as a K1-forward (K2-forward). Similar nomenclature applies to K-calls and K-puts. In preparation for hedging, we also derive the longevity delta and gamma for each of the three securities. As in the previous work of Cairns (2011) and Zhou and Li (2017a, 2017b), the longevity delta and gamma of a mortality-linked security are defined as the first and second partial derivatives of its value with respect to the most recently realised period effects (the CBD mortality indices) respectively. In the end of this section, we study the put-call parity for the securities written on the CBD mortality indices, as well as how the properties of the securities may change with the level of moneyness.

A K-forward is a zero-coupon swap that exchanges on the maturity date a fixed amount for a random amount that is proportional to a CBD mortality index realised at some future time. Consider a K-forward written on the \(i\)th CBD mortality index. Suppose that the K-forward is issued at time \(t_0\) and matures at time \(T\) (where \(T > t_0\)). From the perspective of the fixed rate receiver, the payoff of this K-forward at maturity is \(F^{(i)}(T, K) = K - \kappa^{(i)}\) per $1 notional, where \(K\) represents the fixed leg that is predetermined when the K-forward is issued\(^5\). The value of this K-forward at time \(t\) for \(t_0 \leq t < T\) is the risk-neutral expectation of its payoff \(F^{(i)}(T, K)\), discounted at the risk-free interest rate \(r_f\), given information up to and including time \(t\). That is, \(F^{(i)}(T, K) = E_t^{(Q)}[(1+r_f)^{-(T-t)}F^{(i)}(T, K)] = (1+r_f)^{-(T-t)}(K - E_t^{(Q)}[\kappa^{(i)}]),\) where \(E_t^{(Q)}[\kappa^{(i)}]\) is given by equation (5) according to the model assumptions made.

The time-\(t\) values of the longevity delta and gamma for this K-forward are the first and second order partial derivatives of \(F^{(i)}(T, K)\) with respect to \(\kappa^{(i)}\) respectively. That is,

\[
\Delta^{(i,F)}_t(T, K) = \frac{\partial}{\partial \kappa^{(i)}_t} F^{(i)}_t(T, K) = -(1+r_f)^{-(T-t)}
\]

and

\[
\Gamma^{(i,F)}_t(T, K) = \frac{\partial}{\partial \kappa^{(i)}_t} \Delta^{(i,F)}_t(T, K) = 0
\]

\(^5\) For simplicity, we ignore the potential lag in the availability of the index data.
for $t_0 \leq t < T$. It is not surprising that $\Gamma_t^{(i, F)}(T, K) = 0$ for all $t_0 \leq t < T$ as the payoff is a perfectly linear function of $\kappa_T^{(i)}$. In practice, a K-forward may be constructed in such a way that no cash flow exchanges hands at time $t_0$ when it is issued. To achieve this, we require $F_{0}^{(i)}(T, K) = 0$, which in turn means that the fixed leg $K$ should be set to $E_{0}^{(i)}[\kappa_T^{(i)}]$.

As discussed in Section 3.2, pension plan sponsors are subject to downside $K_1$ and $K_2$ risks. Therefore, to hedge their longevity risk exposures, they may participate in $K_1$- and $K_2$-forwards as a fixed-rate receiver, so that they will receive a net payment from the counterparty to defray their increased pension obligations when the CBD mortality indices turn out to be too low. On the other hand, life insurers, who are typically subject to an upside $K_1$ risk and a downside $K_2$ risk, may hedge their risk exposures by participating as a fixed-rate payer in a $K_1$-forward and a fixed-rate receiver in a $K_2$-forward.

Now consider a (European) K-call on the $i$th CBD mortality index. Suppose that the K-call is issued at time $t_0$, matures at time $T$, and has a strike value of $K$. The payoff of this K-call at maturity is $C^{(i)}(T, K) = \max(\kappa_T^{(i)} - K, 0)$ per $1$ notional, and the price of this K-call at time $t$ for $t_0 \leq t < T$ is

$$
C_t^{(i)}(T, K) = E_Q^{(i)}[(1 + r_f)^{-T-t}C^{(i)}(T, K)]
= (1 + r_f)^{-(T-t)} \left( \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \left( 1 - \phi \left( \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \right) \right) \right) - (K - E_t^{(Q)}[\kappa_T^{(i)}]) \right) \right)
$$

Finally, the time-$t$ values of the longevity delta and gamma for this K-call are

$$
\Delta_t^{(i,C)}(T, K) = \frac{\partial}{\partial \kappa_t^{(i)}} C_t^{(i)}(T, K) = (1 + r_f)^{-(T-t)} \left( 1 - \phi \left( \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \right) \right)
$$

and

$$
\Gamma_t^{(i,C)}(T, K) = \frac{\partial}{\partial \kappa_t^{(i)}} \Delta_t^{(i,C)}(T, K) = (1 + r_f)^{-(T-t)} \frac{1}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \phi \left( \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \right)
$$

respectively. In the above, $E_t^{(Q)}[\kappa_T^{(i)}]$ and $\text{Var}_t^{(Q)}[\kappa_T^{(i)}]$ are given by equations (5) and (6) respectively, according to the model assumptions made. The full derivation of the above equations of $C^{(i)}(T, K)$, $\Delta^{(i,C)}(T, K)$ and $\Gamma^{(i,C)}(T, K)$ is presented in Section 3.9.

Recall that pension plan sponsors are typically subject to downside $K_1$ and $K_2$ risks. As such, they may mitigate their risk exposures by taking a short position in $K_1$- and $K_2$-calls.
Likewise, life insurers, who are typically subject to an upside K1 risk and a downside K2 risk, may reduce their risk exposures by taking a long position in a K1-call and a short position in a K2-call.

Then consider a (European) K-put on the $i$th CBD mortality index. Suppose again that the K-put is issued at time $t_0$, matures at time $T$, and has a strike value of $K$. The payoff of this K-put at maturity is $P^{i}(T, K) = \max(K - \kappa_T^{(i)}, 0)$ per $1$ notional, and the price of this K-put at time $t$ for $t_0 \leq t < T$ is

$$P^{i}_{t}(T, K) = \mathbb{E}^{(Q)}_{t}[(1 + r_f)^{-(T-t)}P^{(i)}(T, K)]$$

$$= (1 + r_f)^{-(T-t)} \left( \frac{K - \mathbb{E}^{(Q)}_{i}[^{\kappa_T^{(i)}]} \Phi \left( \frac{K - \mathbb{E}^{(Q)}_{i}[^{\kappa_T^{(i)}]} \sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]}{\sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]} \right)}{\sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]} \right)$$

$$+ (K - \mathbb{E}^{(Q)}_{i}[^{\kappa_T^{(i)}]} \Phi \left( \frac{K - \mathbb{E}^{(Q)}_{i}[^{\kappa_T^{(i)}]} \sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]}{\sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]} \right))$$

The time-$t$ values of the longevity delta and gamma of this K-put are given by

$$\Delta^{(i,P)}_{t}(T, K) = \frac{\partial}{\partial \kappa_T^{(i)}} P^{(i)}(T, K) = -(1 + r_f)^{-(T-t)} \Phi \left( \frac{K - \mathbb{E}^{(Q)}_{i}[^{\kappa_T^{(i)}]} \sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]}{\sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]} \right)$$

and

$$\Gamma^{(i,P)}_{t}(T, K) = \frac{\partial}{\partial \kappa_T^{(i)}} \Delta^{(i,P)}_{t}(T, K) = (1 + r_f)^{-(T-t)} \frac{1}{\sigma_{t,T}} \Phi \left( \frac{K - \mathbb{E}^{(Q)}_{i}[^{\kappa_T^{(i)}]} \sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]}{\sqrt{\text{Var}^{(Q)}_{i}[^{\kappa_T^{(i)}]}]} \right)$$

respectively. The derivation of $P^{(i)}_{t}(T, K), \Delta^{(i,P)}_{t}(T, K)$ and $\Gamma^{(i,P)}_{t}(T, K)$ is set out in Section 3.9.

Pension plan sponsors, who are usually subject to downside K1 and K2 risks, may reduce their risk exposures by taking a long position in K1- and K2-puts. Life insurers, who are often subject to an upside K1 risk and a downside K2 risk, may mitigate their risk exposures by taking a short position in a K1-put and a long position in a K2-put.

The put-call parity says that a portfolio of a long call option and a short put option is equivalent to (and hence has the same value as) a forward contract at the same strike price (fixed leg) and expiry. This important relationship also holds for options and forwards written on the CBD mortality indices. In terms of payoffs, for fixed $i$, $K$, and $T$, we have $C^{(i)}(T, K) - P^{(i)}(T, K) = \max(\kappa_T^{(i)} - K , 0) - \max(K - \kappa_T^{(i)}, 0) = \kappa_T^{(i)} - K = -f^{(i)}(T, K)$, which corresponds to the per unit notional payoff of the forward contract with parameters $i$, $K$, and $T$ from the fixed rate payer’s perspective. In terms of values, for fixed $i$, $K$, and $T$, we have
\[ C_t^{(i)}(T, K) - P_t^{(i)}(T, K) \]
\[ = (1 + r_f)^{-(T-t)} \left( \sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]} \phi \left( \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \right) - (K - E_t^{(Q)}[\kappa_T^{(i)}]) \left( 1 - \Phi \left( \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \right) \right) \right) \]
\[ = -(1 + r_f)^{-(T-t)} \left( \sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]} \phi \left( \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \right) + (K - E_t^{(Q)}[\kappa_T^{(i)}]) \Phi \left( \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \right) \right) \]
\[ = -F_t^{(i)}(T, K) \]
for \( t_0 \leq t < T \). Note that \(-F_t^{(i)}(T, K)\) corresponds to the per unit notional time-\( t \) value of the forward contract with parameters \( i, K, \) and \( T \) from the fixed rate payer’s perspective.

Consider \( K \)-options with a strike value of \( K \) and maturity date of \( T \). At any time \( t \), a \( K \)-option is called at-the-money if \( K = E_t^{(Q)}[\kappa_T^{(i)}] \). If \( K < E_t^{(Q)}[\kappa_T^{(i)}] \), then the \( K \)-call is called in-the-money while the \( K \)-put is called out-of-the-money. Similarly, if \( K > E_t^{(Q)}[\kappa_T^{(i)}] \), then the \( K \)-put is in-the-money while the \( K \)-call is out-of-the-money. It is obvious that the moneyness of a \( K \)-option depends on the strike value \( K \) and \( E_t^{(Q)}[\kappa_T^{(i)}] \), which further depends on \( \kappa_T^{(i)} \). To facilitate exposition, we define the following moneyness metric:

\[
\text{Moneyness} = \frac{K - E_t^{(Q)}[\kappa_T^{(i)}]}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}}
\]

which measures the difference between \( K \) and \( E_t^{(Q)}[\kappa_T^{(i)}] \) relative to the standard deviation of \( \kappa_T^{(i)} \) given the information up to and including time \( t \). Although the notion of moneyness does not apply to forward contracts, for ease of exposition, in our numerical illustrations we may express the fixed leg \( K \) of a \( K \)-forward in terms of ‘Moneyness’ through equation (8).

When ‘Moneyness’ equals zero (i.e. \( K = E_t^{(Q)}[\kappa_T^{(i)}] \)), the time-\( t \) prices of the \( K \)-call and \( K \)-put take the same value:

\[ C_t^{(i)}(T, K) = P_t^{(i)}(T, K) = (1 + r_f)^{-(T-t)} \sqrt{\frac{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}{2\pi}} \]

This interesting property can be explained as follows. When \( K = E_t^{(Q)}[\kappa_T^{(i)}] \), according to equations (4) and (5), the payoffs of the \( K \)-call and the \( K \)-put under the risk-neutral probability measure are

\[
\max(\kappa_T^{(i)} - K, 0) = \max \left( \sum_{k=1}^{T-t} e_t^{(i)} e_t^{(i)} k, 0 \right)
\]

and
respectively. Since the distribution of \( \sum_{k=1}^{T-t} \xi_{t+k}^{(i)} \) is symmetric, the payoffs from both options are identically distributed, which means that the two options must have the same price.

If the K-call is deeply in-the-money at time \( t \) (i.e. Moneyness → \(-\infty\)), then its price is close to \(-F_t^{(i)}(T, K)\) (i.e. the time-\( t \) value of an otherwise identical K-forward from the fixed rate payer’s perspective). When ‘Moneyness’ becomes very negative, we have \( \phi \left( \frac{K - E_t^{(i)}[\kappa_T^{(i)}]}{\sqrt{\text{var}^{(i)}[\kappa_T^{(i)}]}} \right) \approx 0 \) and \( \Phi \left( \frac{K - E_t^{(i)}[\kappa_T^{(i)}]}{\sqrt{\text{var}^{(i)}[\kappa_T^{(i)}]}} \right) \approx 0 \), so that, according to equation (7), the price of the K-call becomes \( C_t^{(i)}(T, K) \approx -(1+r_f)^{-T-t}(K - E_t^{(i)}[\kappa_T^{(i)}]) = -F_t^{(i)}(T, K) \).

On the other hand, if the K-call is deeply out-of-the-money at time \( t \) (i.e. Moneyness → \(+\infty\)), then its price is close to 0. When ‘Moneyness’ is very large, we have \( \phi \left( \frac{K - E_t^{(i)}[\kappa_T^{(i)}]}{\sqrt{\text{var}^{(i)}[\kappa_T^{(i)}]}} \right) \approx 0 \) and \( \Phi \left( \frac{K - E_t^{(i)}[\kappa_T^{(i)}]}{\sqrt{\text{var}^{(i)}[\kappa_T^{(i)}]}} \right) \approx 1 \), such that the price of the K-call tends to \( C_t^{(i)}(T, K) \approx 0 \).

Using similar arguments, the price of the K-put is \( P_t^{(i)}(T, K) \approx (1+r_f)^{-T-t}(K - E_t^{(i)}[\kappa_T^{(i)}]) = F_t^{(i)}(T, K) \) when it is deeply in-the-money (i.e., Moneyness → \(+\infty\)), and it is \( P_t^{(i)}(T, K) \approx 0 \) when it is deeply out-of-the-money (i.e., Moneyness → \(-\infty\)).

The left panels of Figure 3.3 depict the relationships between moneyness and price, on the basis of the assumed market prices of risk and the parameters that are estimated from real data\(^6\). All of the aforementioned properties can be observed in the diagram. In the middle and right panels of Figure 3.3, we show the relationships between moneyness and the longevity Greeks. The following properties (which can be deduced easily from the formulas presented earlier) can be observed:

1. The longevity delta of a K-forward is insensitive to its strike value \( K \), and the longevity gamma of a K-forward is always 0.
2. At any given time point \( t \), the difference between the longevity deltas of a K-call and a K-put with the same strike value and time-to-maturity \( (T - t) \) is a constant that equals to \((1+r_f)^{-(T-t)}\).
3. At any given time point, the longevity gammas of a K-call and a K-put with the same strike value and time-to-maturity are identical.

\(^6\) Parameter estimation is discussed in Section 3.7.
Figure 3.3 Time-$t$ prices, deltas, and gammas of K-calls, K-puts, and K-forwards with different degrees of moneyness

Note: The values shown are calculated with $r_f = 0.02$, $T - t = 15$, $E_t^{(0)}[\kappa T^{(i)}] = -5.6304$, $\sqrt{\text{Var}}_t^{(0)}[\kappa T^{(i)}] = 6.4874 \times 10^{-2}$, $E_t^{(0)}[\kappa T^{(2)}] = 0.1087$, $\sqrt{\text{Var}}_t^{(0)}[\kappa T^{(2)}] = 4.0693 \times 10^{-3}$, and $K = E_t^{(0)}[\kappa T^{(0)}] + \text{Moneyness} \times \sqrt{\text{Var}}_t^{(0)}[\kappa T^{(0)}]$. 
3.5 Longevity risk exposures

It is assumed that the liability being hedged is a whole life annuity-immediate of $1 that is sold to individuals aged $x_0$ at time $t_0$. Discounted to time $t$ at an effective interest rate of $r$ per annum, the sum of the annuity payments beyond time $t$ (per surviving annuitant at time $t$) is

$$L(t, r) = \sum_{s=1}^{\omega-x_0-t+t_0} (1+r)^{-s} S_{x_0+t-t_0}(s)$$

for $t = t_0, t_0+1, \ldots, t_0+\omega-x_0-1$

where $\omega$ is the limiting age, and

$$S_{x,t}(s) = \prod_{u=1}^{s} (1 + \exp(\beta_{x^u-t-1}^{k_1^{(1)}}))^{-1}$$

with $\beta_t = (1, x-\bar{x}_t)$ represents the ex post probability that an individual aged $x$ at time $t$ would have survived to time $t + s$. The time-$t$ value of the (unpaid) annuity liability is the risk-neutral expectation of $L(t, r)$, given information up to and including time $t$. That is,

$$L_t = E_t^Q[L(t, r)] = \sum_{s=1}^{\omega-x_0-t+t_0} (1+r_j)^{-s} E_t^Q[S_{x_0+t-t_0}(s)]$$

for $t = t_0, t_0+1, \ldots, t_0+\omega-x_0-1$. It is obvious that $L_t$ depends on the values of $E_t^Q[S_{x,t}(s)]$ for $s = 1, 2, \ldots, \omega-x_0-t+t_0$ and $x = x_0+t-t_0$. Furthermore, given the Markov property of the bivariate random walk (under the risk-neutral probability measure), the value of $E_t^Q[S_{x,t}(s)]$ depends exclusively on the values of $k_1^{(1)}$ and $k_1^{(2)}$.

In the following situations, nested simulations are required to estimate the exact values of $L_t$ for $t > t_0$:

1. Dynamic hedging

   For each of the $N$ simulated mortality scenarios (under the real-world probability measure) for evaluating the hedge effectiveness, $M$ simulations (under the risk-neutral probability measure) are needed to calculate $L_t$ for each time point $t$ at which the hedge is adjusted. The entire procedure involves $N \times M \times k$ simulations, where $k$ is the number of time points at which the hedge is adjusted.

2. Evaluation of the $\tau$-year ahead VaR

   To obtain an empirical distribution of $L_t$ from which the $\tau$-year ahead VaR can be estimated, we need $M$ simulations (under the risk-neutral probability measure) to calculate $L_t$ for each of the $N$ simulated mortality scenarios (under the real-world probability measure) for evaluating hedge effectiveness. The entire procedure entails
In the following, we present a method to approximate the values of $E_t^Q[\mathcal{S}_x(t)]$ for $s = 1, 2, \ldots$, $\omega - x_0 - t_0$ and $x = x_0 + t_0$, so that the computationally demanding nested simulations can be avoided. Following Cairns (2011), we apply a first-order Taylor’s series approximation to the probit transform of $E_t^Q[\mathcal{S}_x(t)]$. That is,

$$
\Phi^{-1}(E_t^Q[\mathcal{S}_x(t)]) \approx d_{x,t,0}(s) + d_{x,t,1}(1)(s)(\kappa_t^{(1)} - \hat{k}_t^{(1)}) + d_{x,t,1}(2)(s)(\kappa_t^{(2)} - \hat{k}_t^{(2)})
$$

where $\Phi$ is the standard normal distribution function, $d_{x,t,0}(s) = \Phi^{-1}(E_t^Q[\mathcal{S}_x(t)])$, $d_{x,t,1}(1)(s) = \frac{\partial \Phi^{-1}(E_t^Q[\mathcal{S}_x(t)])}{\partial \kappa_t^{(1)}}$, and $d_{x,t,1}(2)(s) = \frac{\partial \Phi^{-1}(E_t^Q[\mathcal{S}_x(t)])}{\partial \kappa_t^{(2)}}$. The values of $d_{x,t,0}(s)$, $d_{x,t,1}(1)(s)$ and $d_{x,t,1}(2)(s)$ are evaluated at $\kappa_t^{(1)} = \hat{k}_t^{(1)}$ and $\kappa_t^{(2)} = \hat{k}_t^{(2)}$. We set $\hat{k}_t^{(1)}$ and $\hat{k}_t^{(2)}$ to $E_t^{(p)}[\kappa_t^{(1)}]$ and $E_t^{(p)}[\kappa_t^{(2)}]$ respectively, where $E_t^{(p)}[\cdot]$ represents the expectation under the real-world probability measure, given information up to and including time $t$.

To derive $d_{x,t,1}(1)(s)$ and $d_{x,t,1}(2)(s)$, we use the fact that

$$
\frac{\partial \Phi^{-1}(f(x))}{\partial x} = \frac{1}{\phi(\Phi^{-1}(f(x)))} \frac{\partial f(x)}{\partial x}
$$

where $\phi$ is the standard normal probability density function. It follows that

$$
d_{x,t,1}(i)(s) = \frac{1}{\phi(\Phi^{-1}(E_t^Q[\mathcal{S}_x(t)])}) \left( \frac{\partial \Phi^{-1}(E_t^Q[\mathcal{S}_x(t)])}{\partial \kappa_t^{(i)}} \frac{\partial E_t^Q[\mathcal{S}_x(t)]}{\partial \kappa_t^{(i)}} + \frac{\partial E_t^Q[\mathcal{S}_x(t)]}{\partial \kappa_t^{(i)}} \sum_{u=1}^{s-1} (1 + \exp(\beta_{x+u-1}\kappa_t^{(i)})) \right)
$$

In the above, $\frac{\partial}{\partial \kappa_t^{(i)}} \prod_{u=1}^{s} (1 + \exp(\beta_{x+u-1}\kappa_t^{(i)}))$ can be calculated recursively as

$$
\frac{\partial}{\partial \kappa_t^{(i)}} \prod_{u=1}^{s} (1 + \exp(\beta_{x+u-1}\kappa_t^{(i)})) = \begin{cases} 
\frac{\partial \beta_{x+1}\kappa_t^{(i)}}{\partial \kappa_t^{(i)}} \exp(\beta_{x+1}\kappa_t^{(i)}), & s = 1 \\
\frac{\partial \beta_{x+s-1}\kappa_t^{(i)}}{\partial \kappa_t^{(i)}} \exp(\beta_{x+s-1}\kappa_t^{(i)}) \prod_{u=1}^{s-1} (1 + \exp(\beta_{x+u-1}\kappa_t^{(i)})) + (1 + \exp(\beta_{x+s-1}\kappa_t^{(i)})) \frac{\partial}{\partial \kappa_t^{(i)}} \prod_{u=1}^{s-1} (1 + \exp(\beta_{x+u-1}\kappa_t^{(i)})), & s > 1 
\end{cases}
$$

(9)
where, according to equation (4),
\[
\frac{\partial \beta_{x+s-1} \kappa_t}{\partial \kappa_t^{(1)}} = \frac{\partial \kappa_t^{(1)}}{\partial \kappa_t^{(1)}} + (x + s - 1 - \bar{x}) \frac{\partial \kappa_t^{(2)}}{\partial \kappa_t^{(1)}} = 1
\]
and
\[
\frac{\partial \beta_{x+s} \kappa_t}{\partial \kappa_t^{(2)}} = \frac{\partial \kappa_t^{(1)}}{\partial \kappa_t^{(2)}} + (x + s - 1 - \bar{x}) \frac{\partial \kappa_t^{(2)}}{\partial \kappa_t^{(2)}} = x + s - 1 - \bar{x}
\]
for \( s \geq 1 \).

The longevity deltas of the (unpaid) annuity liability are defined as the first partial derivatives of \( L_t \) with respect to \( \kappa_t^{(1)} \) and \( \kappa_t^{(2)} \). That is,
\[
\Delta_t^{(i,L)} = \frac{\partial L_t}{\partial \kappa_t^{(i)}} = \sum_{s=1}^{\omega-x_0-t_0} (1 + rf)^{-s} \frac{\partial}{\partial \kappa_t^{(i)}} E_t^{(Q)} [S_{x_0+t-t_0}(s)], \quad i = 1, 2
\]
To determine \( \Delta_t^{(i,L)} \), we need to compute the values of \( \frac{\partial}{\partial \kappa_t^{(i)}} E_t^{(Q)} [S_{x,t}(s)] \) for \( s = 1, 2, \ldots, \omega-x_0+t_0 \) and \( x = x_0+t-t_0 \). When \( t = t_0 \), we have
\[
\frac{\partial}{\partial \kappa_t^{(i)}} E_t^{(Q)} [S_{x,t}(s)] = E_t^{(Q)} \left( (S_{x,t}(s))^{2} \frac{\partial}{\partial \kappa_t^{(i)}} \prod_{u=1}^{s} (1 + \exp(\beta_{x+u-1} \kappa_{t+u})) \right), \quad i = 1, 2
\]
where \( \frac{\partial}{\partial \kappa_t^{(i)}} \prod_{u=1}^{s} (1 + \exp(\beta_{x+u-1} \kappa_{t+u})) \) can be obtained using equation (9). When \( t > t_0 \), we use the approximation formula to get
\[
\frac{\partial}{\partial \kappa_t^{(i)}} E_t^{(Q)} [S_{x,t}(s)] \approx \frac{\partial}{\partial \kappa_t^{(i)}} \Phi \left( d_{x,t,0}(s) + d_{x,t,1}(s)(\kappa_t^{(1)} - \kappa_t^{(1)}(1)) + d_{x,t,1}(s)(\kappa_t^{(2)} - \kappa_t^{(2)}(1)) \right)
\]
\[
= \Phi \left( d_{x,t,0}(s) + d_{x,t,1}(s)(\kappa_t^{(1)} - \kappa_t^{(1)}(1)) + d_{x,t,1}(s)(\kappa_t^{(2)} - \kappa_t^{(2)}(1)) \right) d_{x,t,1}(s)
\]
for \( i = 1, 2 \).

We now evaluate the accuracy of the approximation method discussed above. The evaluation is based on a (real-world) Cairns-Blake-Dowd model that is fitted to historical data up to and including year \( t_0 = 2013 \) and its risk-neutral counterpart that is defined using \( \lambda^{(1)} = \lambda^{(2)} = 0.175 \) (see Section 3.7 for further details). Figure 3.4 demonstrates the accuracy of the approximation for \( L_{t_0+1} \), a quantity that is highly relevant to the calculation of the 1-year VaR and the evaluation of an annually adjusted dynamic hedge\(^7\). In both panels, the dots represent 1,000 simulated pairs of \( (\kappa_{t_0+1}^{(1)}, \kappa_{t_0+1}^{(2)}) \), so the cloud of dots can be understood as the possible range of \( (\kappa_{t_0+1}^{(1)}, \kappa_{t_0+1}^{(2)}) \). In the left panel, the solid blue lines represent the ‘true’ values which are calculated by full (nested) simulations, whereas the dashed red lines

\(^7\) The evaluation results for \( L_{t_0+i} \), \( i = 2, 3, \ldots \) are similar and therefore not shown.

\(^8\) The 1,000 pairs of \( (\kappa_{t_0+1}^{(1)}, \kappa_{t_0+1}^{(2)}) \) are simulated under the real-world probability measure.
represent the approximated values. It can be observed that the gaps between the solid and dashed lines are very small, indicating that the approximation is very accurate. In the right panel, the contour lines represent the absolute percentage errors in approximating \( L_{t,0+1} \). Within the boundary of the cloud of dots, the absolute percentage errors are no greater than 0.03\%, further confirming that the approximation is highly accurate. The small absolute percentage errors also suggest that a higher-order approximation does not seem to be necessary.

**Figure 3.4** An illustration of the accuracy of the approximation for \( L_{t,0+1} \)

![Image](image.png)

Note: In the left panel, contours represent the true (solid blue) and approximated (dashed red) values of \( L_{t,0+1} \). In the right panel, contours represent the absolute percentage errors of the approximation.

In principle, we can apply similar approximations to the values of K-options, although such approximations are not necessary as exact analytical valuation formulas are available (see Section 3.4). For instance, when using a second-order approximation, the time-\( t \) value of a K-put can be approximated as

\[
P_t^{(i)}(T,K) \approx c_{x,t,0}(s) + c_{x,t,1}^{(i)}(s)(\kappa_t^{(i)} - \bar{\kappa}_t^{(i)}) + c_{x,t,2}^{(i)}(s)(\kappa_t^{(i)} - \bar{\kappa}_t^{(i)})^2
\]

where

\[
c_{x,t,0}(s) = E_t^{(Q)}[(1 + r_f)^{-(T-t)} \max(K - \kappa_T^{(i)}, 0)]
\]
\[ c^{(i)}_{x,t,1}(s) = \frac{\partial}{\partial \kappa^{(i)}_t} E^{(Q)}_t \left[ (1 + r_f)^{-(T-t)} \max(K - \kappa^{(i)}_T, 0) \right] \]
\[ = -(1 + r_f)^{-(T-t)} \Phi \left( \frac{K - E_t^{(Q)}[\kappa^{(i)}_T]}{\sqrt{\text{Var}_t^{(Q)}[\kappa^{(i)}_T]}} \right) \]
\[ = \Delta^{(i,p)}_t(T, K) \]

and
\[ c^{(i)}_{x,t,2}(s) = \frac{\partial^2}{\partial (\kappa^{(i)}_t)^2} E^{(Q)}_t \left[ (1 + r_f)^{-(T-t)} \max(K - \kappa^{(i)}_T, 0) \right] \]
\[ = (1 + r_f)^{-(T-t)} \frac{1}{\sigma_{t,T}} \phi \left( \frac{K - E_t^{(Q)}[\kappa^{(i)}_T]}{\sqrt{\text{Var}_t^{(Q)}[\kappa^{(i)}_T]}} \right) \]
\[ = \Gamma^{(i,p)}_t(T, K) \]

with \( c_{x,t,0}(s), c_{x,t,1}(s), \) and \( c_{x,t,2}(s) \) being calculated at \( \kappa^{(1)}_t = \kappa^{(1)}_0 \) and \( \kappa^{(2)}_t = \kappa^{(2)}_0 \).

Figure 3.5 demonstrates the accuracy of the approximation for \( P^{(1)}_{t_0 + 15}(t_0 + 15, E^{(Q)}_t[\kappa^{(1)}_{t_0 + 15}]) \), the time-\((t_0+1)\) value of a K-put on the first CBD mortality index with a maturity date of \( t_0 + 15 \) and a strike value of \( E^{(Q)}_t[\kappa^{(1)}_{t_0 + 15}] \) (the option is at-the-money when it was issued at time \( t_0 \)). In the figure, we show the absolute percentage errors (relative to the true values calculated from the analytical valuation formula) incurred in the linear approximation (left panel) and the quadratic approximation (right panel), along with the empirical distribution of \( \kappa_{t_0+1}^{(1)} \) given information up to and including time \( t_0 \) under the real-world probability measure. By definition, the approximation is exact when \( \kappa_{t_0+1}^{(1)} = \kappa^{(1)}_{t_0} \).

As \( \kappa_{t_0+1}^{(1)} \) moves away from \( \kappa^{(1)}_{t_0} \), the absolute percentage error increases rapidly. For the linear approximation, the absolute percentage errors are larger than 100% for some simulated values of \( \kappa_{t_0+1}^{(1)} \). The quadratic approximation is somewhat more accurate, but the absolute percentage errors may still be larger than 10% in some circumstances. The results shown in Figure 3.5 confirm that Taylor’s series approximations do not perform satisfactorily for the values of mortality derivatives with an option-like payoff, and it highlights the importance of the exact analytical valuation formulas provided in Section 3.4.
Figure 3.5: Absolute percentage errors resulting from linear (left panel) and quadratic (right panel) approximations of $P_{t_0+1}^{(1)}(t_0 + 15, E_{t_0}^{(Q)}[k_{t_0+1}^{(1)}])$

Note: The histogram represents the empirical distribution of $k_{t_0+1}^{(1)}$ given the information up to and including time $t_0$ under the real-world probability measure.
3.6 Developing longevity hedges

In this section, we develop delta longevity hedges using K-forwards and K-options. It is assumed that at any time, the hedge contains two instruments, one written on each of the two CBD mortality indices.

All longevity hedges are assumed to be established at time \( t_0 \). Regardless of how the longevity hedge is formed, the hedge effectiveness is measured in terms of the VaR, estimated using mortality scenarios that are simulated under the real-world probability measure (given information up to and including time \( t_0 \)). Accordingly, all simulated cash flows should be discounted using the hedger’s risk-adjusted interest rate \( r_a \) instead of the risk-free interest rate \( r_f \). Given the assumed value of \( r_f \), the value of \( r_a \) is obtained by solving the following equation numerically:

\[
E_{t_0}^{(0)}[L(t_0, r_f)] = E_{t_0}^{(P)}[L(t_0, r_a)]
\]

We start with cash-flow hedges, which focus on the variability of the (discounted) cash flows arising from the annuity liability and the hedging instruments. Without any longevity hedge in place, the sum of all annuity payments discounted to time \( t_0 \) at the risk-adjusted interest rate of \( r_a \) is simply \( L(t_0, r_a) \). We can straightforwardly simulate an empirical distribution of \( L(t_0, r_a) \) under the real-world probability measure. From the empirical distribution of \( L(t_0, r_a) \), we can obtain the VaR at the 99.5 confidence level (i.e. the 99.5th percentile), which is then used to benchmark against the VaR values of the hedged positions.

In a static hedge, no adjustment is made after time \( t_0 \) when the hedge is established. For the \( i \)th instrument in the hedge, we use \( T(i) \) to represent its maturity date, and \( K(i) \) to represent its strike value (if the instrument is a K-option) or its fixed leg (if the instrument is a K-forward). The notional amounts of the two instruments are determined by matching the longevity deltas of the hedge portfolio and the liability being hedged. It follows that the notional amount of the \( i \)th instrument is given by

\[
h(i) = \frac{\Delta_{t_0}^{(i,L)}}{\Delta_{t_0}^{(i,H)}(T(i), K(i))}, \quad i = 1, 2
\]

where \( H = F, C, \) or \( P \) depending on whether the \( i \)th instrument is a K-forward, K-call, or K-put. The hedging instruments incur a cash outflow (associated with their purchase prices) of
\[ \sum_{i=1}^{2} h^{(i)} H_{t_0}^{(i)}(T^{(i)}, K^{(i)}) \]

at time \( t_0 \), and a cash inflow (associated with their payoffs) of
\[ \sum_{i=1}^{2} h^{(i)} \mathcal{H}^{(i)}(T^{(i)}, K^{(i)}) \]

where \( \mathcal{H} = F, C, \) or \( P \) depending on whether the \( i \)th instrument is a K-forward, K-call, or K-put. As a result, discounted to time \( t_0 \) at the risk-adjusted interest rate, the net cash outflows arising from the hedging instrument sum to
\[ \sum_{i=1}^{2} h^{(i)} \left( H_{t_0}^{(i)}(T^{(i)}, K^{(i)}) - (1 + r_a)^{-(T^{(i)} - t_0)} \mathcal{H}^{(i)}(T^{(i)}, K^{(i)}) \right) \]

(11)

The performance of the hedge can be evaluated using the VaR at the 99.5% confidence level, obtained from the 99.5th percentile of the distribution of
\[ \mathcal{L}(t_0, r_a) + \sum_{i=1}^{2} h^{(i)} \left( H_{t_0}^{(i)}(T^{(i)}, K^{(i)}) - (1 + r_a)^{-(T^{(i)} - t_0)} \mathcal{H}^{(i)}(T^{(i)}, K^{(i)}) \right) \]

(12)

which represents the total discounted cash outflows of the hedged position. The empirical distribution of (12) can be obtained using simulated values of \( \mathcal{L}(t_0, r_a) \) and \( \mathcal{H}^{(i)}(T^{(i)}, K^{(i)}) \), given the information up to and including time \( t_0 \), under the real-world probability measure. The values of \( h^{(i)} \) and \( H_{t_0}^{(i)}(T^{(i)}, K^{(i)}) \) in expression (12) should be calculated using the exact formulas presented in Sections 3.4 and 3.5 prior to the simulation.

In a dynamic hedge, the hedge portfolio is adjusted annually until the annuity liability runs off completely. Specifically, it is assumed that at each time point \( t \), for \( t = t_0 + 1, \ldots, t_0 + \omega - x_0 - 1 \), the two hedging instruments purchased at the previous time point \( (t - 1) \) are closed out, and another two hedging instruments are purchased to suit the current longevity risk profile of the annuity liability. For the \( i \)th hedging instrument purchased at time \( t \), where \( t = t_0, \ldots, t_0 + \omega - x_0 - 1 \), we use \( T_t^{(i)} \) to denote its maturity date and \( K_t^{(i)} \) to represent its strike value. The notional amounts of the hedging instruments purchased at time \( t \) are determined by matching the time-\( t \) values of the longevity deltas of the annuity liability and the hedging instruments. This strategy implies that the notional amount of the \( i \)th instrument purchased at time \( t \) is
\[ h_t^{(i)} = \frac{\Delta_t^{(i,L)}}{\Delta_t^{(i,H)}} \left( T_t^{(i)}(i), K_t^{(i)} \right) \]

for \( t = t_0, \ldots, t_0 + \omega - x_0 - 1, i = 1, 2 \), and \( H = F, P, \) or \( C \). The hedging instruments purchased at
time $t$ incur a cash outflow (which is associated with their purchase prices) of
\[
\sum_{i=1}^{2} h_t^{(i)} H_t^{(i)}(T_t^{(i)}, K_t^{(i)})
\]
at time $t$ and a cash inflow (which is associated with the prices for which they are sold) of
\[
\sum_{i=1}^{2} h_t^{(i)} H_{t+1}^{(i)}(T_t^{(i)}, K_t^{(i)})
\]
when they are being closed out at time $t + 1$. As such, the net cast outflow, discounted to time $t$ at the risk-adjusted interest rate, arising from the hedging instruments purchased at time $t$ is
\[
\sum_{i=1}^{2} h_t^{(i)} \left( H_t^{(i)}(T_t^{(i)}, K_t^{(i)}) - (1 + r_a)^{-1} H_{t+1}^{(i)}(T_t^{(i)}, K_t^{(i)}) \right)
\]
Consequently, the total net cash outflow, discounted to time $t_0$ at the risk-adjusted interest rate, arising from all the hedging instruments involved in the dynamic hedge is
\[
\sum_{t=t_0}^{t_0+\omega-x_0-1} (1 + r_a)^{-(t-t_0)} \sum_{i=1}^{2} h_t^{(i)} \left( H_t^{(i)}(T_t^{(i)}, K_t^{(i)}) - (1 + r_a)^{-1} H_{t+1}^{(i)}(T_t^{(i)}, K_t^{(i)}) \right)
\] (13)
We may therefore evaluate the performance of the dynamic hedge using the VaR at the 99.5% confidence level (the 99.5th percentile) of the distribution of
\[
\mathcal{L}(t_0, r_a) + \sum_{t=t_0}^{t_0+\omega-x_0-1} (1 + r_a)^{-(t-t_0)} \sum_{i=1}^{2} h_t^{(i)} \left( H_t^{(i)}(T_t^{(i)}, K_t^{(i)}) - (1 + r_a)^{-1} H_{t+1}^{(i)}(T_t^{(i)}, K_t^{(i)}) \right),
\] (14)
which represents the total discounted cash outflows of the hedged position. The empirical distribution of (14) can be obtained using the following procedure:

1. Calculate the values of $h_0^{(i)}$ and $H_0^{(i)}(T_0^{(i)}, K_0^{(i)})$ using the exact formulas presented in Sections 3.4 and 3.5.

2. Simulate a large number of mortality scenarios, given the information up to and including time $t_0$, under the real-world probability measure. For each scenario, perform the following calculations:
   (a) Calculate the realised value of $\mathcal{L}(t_0, r_a)$.
   (b) Calculate the realised values of $H_t^{(i)}(T_t^{(i)}, K_t^{(i)})$ for $t = t_0 + 1, \ldots, t_0 + \omega - x_0 - 1$ and $H_{t+1}^{(i)}(T_t^{(i)}, K_t^{(i)})$ for $t = t_0, \ldots, t_0 + \omega - x_0 - 1$, using the exact formulas presented in Section 3.4.
   (c) Calculate the realised values of $h_t^{(i)}$ for $t = t_0 + 1, \ldots, t_0 + \omega - x_0 - 1$, using the exact formulas presented in Section 3.4 (for the denominator of $h_t^{(i)}$) and the approximation.
formula in Section 3.5 (for the numerator of $h^{(i)}$).

3. The previous steps yield a large number of realisations of (14), from which an empirical distribution of (14) can be obtained.

We now turn to value hedges, which focus on the variability of the values of the hedged position at a certain future time point, say $\tau$ years from time $t_0$. In the following presentation, we consider particularly the situation when $\tau = 1$, which is the most relevant to typical capital requirements. At time $t_0 + 1$, for each of those who are alive, a payment of $1 is made and the value of the remaining liability becomes $L_{t_0+1}$. There is no payment or financial obligation to those who died between time $t_0$ and $t_0 + 1$. Consequently, measured in time-$t_0$ dollars, the value of the unhedged position at time $t_0 + 1$ is given by

$$\tag{15} (1+r_a)^{-1}(S_{t_0,t_0}(1) + S_{0,t_0}(1)L_{t_0+1})$$

The VaR at the 99.5% confidence level (the 99.5th percentile) of the distribution of (15), given information up to and including time $t_0$, under the real-world probability measure can be used as a benchmark for the performance of the value hedges.

As in a static cash-flow hedge, the notional amount of the $i$th hedging instrument in a (static) value hedge is given by

$$h^{(i)} = \frac{\Delta^{(i,L)}_{t_0}}{\Delta^{(i,H)}_{t_0}(T^{(i)}, K^{(i)})}$$

for $i = 1, 2$ and $H = F, C$, or $P$.

Measured in time-$t_0$ dollars, the change in the total value of the two hedging instruments is given by

$$\sum_{i=1}^{2} h^{(i)} \left((1 + r_a)^{-1}H_{t_0+1}^{(i)}(T^{(i)}, K^{(i)}) - H_{t_0}^{(i)}(T^{(i)}, K^{(i)})\right)$$

As a result, the $t_0$-value of the hedged position, measured in time-$t_0$ dollars, is

$$\tag{17} (1+r_a)^{-1}S_{t_0,t_0}(1) + S_{0,t_0}(1)L_{t_0+1} - \sum_{i=1}^{2} h^{(i)} \left((1 + r_a)^{-1}H_{t_0+1}^{(i)}(T^{(i)}, K^{(i)}) - H_{t_0}^{(i)}(T^{(i)}, K^{(i)})\right).$$

We may hence assess the performance of the value hedge using the VaR at the 99.5% (the 99.5th percentile) of the distribution of (17), given the information up to and including $t_0$, under the real-world probability measure.

The following procedure is used to generate the empirical distributions of (15) and (17), given the information up to and including time $t_0$, under the real-world probability measure:
1. Calculate the values of $h^{(i)}$ and $H_{0}^{(i)}(T^{(i)}, K^{(i)})$ using the exact formulas presented in Sections 3.4 and 3.5.

2. Simulate a large number of mortality scenarios under the real-world probability measure. For each scenario, perform the following calculations:
   (a) Calculate the realised value of $s_{0,0}(1)$.
   (b) Calculate the realised values of $H_{0+1}^{(i)}(T^{(i)}, K^{(i)})$ using the exact valuation formulas shown in Section 3.4.
   (c) Calculate the realised values of $L_{0+1}$ using the approximation method described in Section 3.5.

3. The previous steps yield a large number of realisations of (15) and (17), from which empirical distributions of (15) and (17) can be obtained.
3.7 Numerical illustrations

The illustrations in this section are based on the historical data from the English and Welsh male population, with an age range of 40 to 90 and a calibration window of 1964 to 2013. We fit the Cairns-Blake-Dowd model (under the real-world probability measure) to the dataset under consideration, using the method of least squares as in the original work of Cairns et al. (2006b). Note that the chosen age range implies $\bar{x} = 65$. The estimated values of the two CBD mortality indices $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ over the calibration window are displayed in Figure 3.6 (solid lines). A bivariate random walk with drift is then used to capture the dynamics of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$. The estimates of the parameters $\mu$ and $A$ in the process are

$$
\begin{pmatrix}
-1.7346 \times 10^{-2} \\
2.0549 \times 10^{-5}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
2.1354 \times 10^{-2} \\
0
\end{pmatrix}
\begin{pmatrix}
4.8559 \times 10^{-4} \\
7.2807 \times 10^{-4}
\end{pmatrix}
$$

respectively.

As mentioned in Section 3.3, in the baseline calculations we assume that the vector of market prices of risk is $\lambda = (0.175, 0.175)'$. This assumption implies that under the risk-neutral probability measure, the dynamics of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ follows a bivariate random walk with an adjusted drift vector of $\tilde{u} = (-2.1167 \times 10^{-2}, -1.0686 \times 10^{-4})'$. The fan charts in Figure 3.6 compare the simulated sample paths of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ under the real-world and risk-neutral probability measures.

Figure 3.6 Estimated values of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ for $t = 1964, \ldots, 2013$ (solid lines), and simulated values of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ for $t > 2013$ under real-world and risk-neutral probability measures (fan charts)
The following assumptions are made in our baseline illustrations:

- Three types of index-based hedges are considered: (a) static cash-flow hedges, (b) dynamic cash-flow hedges, and (c) static value hedges (with a horizon of one year).
- All hedges are established at \( t_0 = 2013 \), the end point of the calibration window.
- The annuity liability is sold to individuals aged \( x_0 = 65 \) at time \( t_0 = 2013 \). The limiting age of the annuitants is assumed to be \( \omega = 100 \).
- For all types of hedges, the composition of the hedge portfolio is either (a) one K-forward (written as a fixed-rate receiver) on each of the two CBD mortality indices, or (b) one K-put on each of the two CBD mortality indices. In all the hedges, the ‘Moneyness’ of the two instruments used are identical.
- The time-to-maturity of all hedging instruments (measured at the time when they are purchased) is 15 years.
- Both the CBD mortality indices and the annuity liability are linked to the mortality experience of the English and Welsh male population.
- The vector of market prices of risk is \( \vec{\lambda} = (0.175, 0.175)' \).
- The risk-free interest rate is \( r_f = 0.02 \) per annum. The corresponding risk-adjusted interest rate \( (r_a) \) is calculated using equation (10).

The baseline results for different degrees of ‘Moneyness’ are shown graphically in Figure 3.7. Let us first focus on the static cash-flow hedges (the left panel). For the unhedged position, the VaR at the 99.5% confidence level is 17.44. Regardless of its ‘Moneyness’ (i.e. the value of the fixed leg \( K \)), a K-forward hedge can reduce the VaR to a value that is significantly closer to the mean \( (L_{2013} = 16.04) \). Also, the VaR it produces reduces (gently) linearly with its fixed leg \( K \). This relationship can be explained as follows. When the fixed leg \( K \) equals \( E_0^{(Q)}[\kappa T^{(i)}] \), no cash flow exchanges hands at time \( t_0 \) (see Section 3.4), so that the entire hedge cost is payable at maturity when the K-forward is settled. As the fixed leg \( K \) is higher than \( E_0^{(Q)}[\kappa T^{(i)}] \), the amount payable to the counterparty at time \( t_0 \) becomes positive, so that at least part of the cost of the K-forward is payable at time \( t_0 \) instead of at maturity. Because \( r_a < r_f \) (given positive market prices of risk), costs payable at time \( t_0 \) would be lower than that payable at maturity when the time value of money and the interest differential are taken into account, thereby dragging the VaR of the hedged position downwards. Note that

\[9\] ‘Moneyness’ is defined in equation (8).
when \( r_a = r_f \), the VaR produced by the K-forward hedges would be completely insensitive to the fixed leg \( K \) (further illustrations are provided below).

**Figure 3.7** Baseline results of static cash-flow hedges (left panel), dynamic cash-flow hedges (middle panel), and static value hedges (right panel) at different degrees of ‘Moneyness’

The VaR values of the K-forward and K-put (static cash-flow) hedges converge as ‘Moneyness’ becomes very high. This result is expected, because, as explained in Section 3.4, a K-put behaves very similarly to its corresponding K-forward as ‘Moneyness’ approaches positive infinity. In this particular setup, the K-put hedges do not outperform the corresponding K-forward hedges for most degrees of ‘Moneyness’. To explain this, let us consider the following two factors that determine the relative performance between K-put and K-forward hedges:

I. **Cost of hedging**

   It is conceivable that a K-put hedge is less costly than the corresponding K-forward hedge, because the hedger needs to pay for only the downside risk when he / she uses a K-put. This fact can be observed in the left panel of Figure 3.8, which compares the expected costs of hedging for the K-put and K-forward static cash-flow hedges.\(^{10}\)

II. **Effectiveness as a hedging instrument**

   As demonstrated in the middle panel of Figure 3.3, the delta of a K-put is always smaller in magnitude than that of the corresponding K-forward. As such, the hedge ratio (the notional amount) of a K-put hedge is always larger in magnitude than that of

---

\(^{10}\) For a static cash-flow hedge, the expected cost of hedging is calculated as the real-world expected value of (11) given information up to and including time \( t_0 \).
the corresponding K-forward hedge. Hence, compared to a K-forward hedge, a K-put hedge is more effective in the sense that it will pay a larger payoff to the hedger in an adverse scenario.

Figure 3.8  Expected costs of hedging for baseline static cash-flow hedges (left panel), dynamic cash-flow hedges (middle panel), and static value hedges (right panel) at different degrees of ‘Moneyness’

To illustrate the interaction between these two offsetting factors, let us analyse the K-forward and K-put static cash-flow hedges in more detail. In each panel of Figure 3.9, 10,000 red dots (for the K-put hedge) and 10,000 blue dots (for the K-forward hedge) are shown. The \( x \)-coordinate of each dot corresponds to a simulated value of the unhedged position \( L(t_0, r_a) \), whereas the \( y \)-coordinate represents the corresponding simulated value of the hedged position (specified by (12)). The dots shown can be interpreted as follows:

- In a simulated scenario when a hedge takes no action, the dot associated with the scenario and the hedge should lie on a 45 degree line\(^\text{11}\).
- In a simulated scenario when a hedge benefits the hedger, the dot associated with the scenario and the hedge should lie below the 45 degree line, and vice versa.
- The VaR of the hedged position is the vertical position of the 500th dot from top to bottom. The VaR of each hedge is shown as a horizontal solid line in the diagram.
- The clouds of blue dots are symmetric, due to the K-forwards’ symmetric payoff structures. The height of each cloud of blue dots is smaller than the width, suggesting that a K-forward hedge reduces the general dispersion of portfolio values. For many of the blue dots on the left, the \( y \)-coordinate is higher than the \( x \)-coordinate. This

\[^{11}\] The 45 degree line may not pass through the origin due to a non-zero hedge cost.
observation suggests that the hedger has to sacrifice his/her upside potential when he/she chooses to use a K-forward hedge.

- When ‘Moneyness’ becomes very negative, all red dots lie on a 45 degree line since the K-puts never yield a payoff. In the other extreme, the clouds of red and blue dots tend to overlap each other, because, as discussed in Section 3.4, a K-put converges to a K-forward as ‘Moneyness’ becomes very high.

**Figure 3.9** Scattered plots showing the actions of K-forward and K-put hedges with different degrees of ‘Moneyness’ in 10,000 simulated mortality scenarios

We observe that as ‘Moneyness’ increases from −3 to a less negative value, the K-put hedge is effective in the sense that it significantly lowers the positions of a good proportion of red dots. However, this effectiveness is not quite manifested in the vertical position of the 500th dot (top to bottom). As such, at those levels of moneyness, the K-put hedge does not yield a lower VaR at the 99.5% confidence level than the K-forward hedge. When ‘Moneyness’ equals 1.5, the vertical positions of the highest red dots become lower than those of the highest blue dots, so that the K-put hedge results in a lower VaR at the 99.5% confidence level.
confidence level than the K-forward hedge. We emphasise that the interaction between the two factors depends heavily on the setup and parameters. In some other circumstances, a K-put hedge outperforms the corresponding K-forward hedge for a much wider range of ‘Moneyness’.

When ‘Moneyness’ is very negative, a static cash-flow K-put hedge yields a VaR that is even higher than that of the unhedged position. This outcome arises because when ‘Moneyness’ is very negative (i.e. the strike value $K$ is very small), the price of a K-put is low but still non-zero, yet none of the simulated CBD mortality indices falls below the strike value so that the K-put never takes any action in offsetting adverse financial outcomes.

Next, we turn to the dynamic cash-flow hedges (the middle panel of Figure 3.7). The comments made on the static cash-flow hedges also apply to the dynamic cash-flow hedges. Strikingly, in contrast to what was found in the previous studies by Cairns (2011) and Zhou and Li (2017a), we observe that the dynamic cash-flow hedges are only moderately more effective than their static counterpart in terms of reduction in the VaR at the 99.5% confidence level$^{12}$. This outcome may be attributed to the empirical fact that a dynamic hedge is substantially more costly than a static hedge (see the left and middle panels of Figure 3.8), and the increase in the cost of hedging offsets the risk reduction benefits arising from the periodic adjustments$^{13}$.

Finally, we consider the (one-year) static value hedges (the right panel of Figure 3.7). Regardless of whether K-puts or K-forwards are used, a value hedge brings the VaR at the 99.5% confidence level down to a value that is almost equal to the mean ($L_{2013} = 16.04$). The remarkable reduction in the VaR may be attributed to the fact that the one-year value hedge requires the hedger to hold the hedging instruments for only one year, so that the expected cost of a value hedge is significantly lower compared to the corresponding cash-flow hedges (see the right panel of Figure 3.8)$^{14}$.

In the rest of this section, we examine the sensitivity of the hedging results to changes in the (a) market prices of risk, (b) times-to-maturity of the hedging instruments, and (c) risk-free interest rate.

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$^{12}$ Cairns (2011) and Zhou and Li (2017a) did not incorporate the costs of hedging into their results.

$^{13}$ For a dynamic cash-flow hedge, the expected cost of hedging is computed as the real-world expected value of (13) given information up to and including time $t_0$.

$^{14}$ For a value hedge, the expected cost of hedging is calculated as the negative of the real-world expected value of (16) given information up to and including time $t_0$. 
We now consider three alternative sets of market prices of risk, \((0, 0)\)', \((0.3, 0.3)\)', and \((0.5, 0.5)\)', while retaining all the other baseline assumptions. In the extreme case when \(\lambda = (0, 0)'\), the K-put and K-forward hedges are costless. The results based on the alternative sets of market prices of risk are displayed in Figure 3.10. When \(\lambda = (0, 0)'\), we have \(r_a = r_f\) and in the absence of the interest rate differential, the result of the K-forward hedge is completely insensitive to its ‘Moneyness’ (the fixed leg \(K\)). Also, when \(\lambda = (0, 0)'\), all hedges are costless, and therefore the benefit of dynamically adjusting a hedge to the reduction in the VaR becomes more apparent.

Figure 3.10  Performance of static cash-flow hedges (left panel), dynamic cash-flow hedges (middle panel), and static value hedges (right panel) at different degrees of ‘Moneyness’, when three alternative sets of market prices of risk, \((0, 0)\)', \((0.3, 0.3)\)', and \((0.5, 0.5)\)', are considered.
As the market prices of risk increase, the cost advantage of a K-put hedge over the corresponding K-forward hedge becomes more significant. Consequently, when the market prices of risk are increased from 0.175 (the baseline assumption) to 0.3, a K-put hedge yields a lower VaR than the corresponding K-forward hedge for a wider range of ‘Moneyness’. When the market prices of risk are further increased to 0.5, a K-put hedge always results in a lower VaR at the 99.5% confidence level than the corresponding K-forward hedge; however, at such high market prices of risk, the VaR of the hedged position (regardless of which instruments are used) is always higher than that of the unhedged position, meaning that a longevity hedge is no longer economically justifiable.

Finally, we observe that for all the values of $\lambda$ under consideration, the (one-year) value hedges can always bring the VaR at the 99.5% confidence level to a value that is very close to the mean liability value. The insensitivity to the market prices of risk may again be attributed to the fact that in such hedges, the hedger needs to hold the hedging instruments for only one year, so that the impact of a change in the market prices of risk is rather modest.

We now consider three alternative times-to-maturity, 10, 20, and 30 years, while retaining all the other baseline assumptions. The results based on the alternative times-to-maturity are shown in Figure 3.11. Several noteworthy observations can be made. First, for the static cash-flow hedges, as the times-to-maturity lengthen, the range of ‘Moneyness’ over which the red dashed line (representing the VaR produced by the K-put hedge) lies below the solid blue line (representing the VaR produced by the K-forward hedge) becomes wider. This outcome occurs because when the times-to-maturity rise, all the hedging instruments become more costly and hence the cost advantage of K-puts over K-forwards becomes more
significant. Second, for the static cash-flow hedges, when the times-to-maturity are increased to 30 years, the K-forward hedge is never economically justifiable and the K-put hedge is economically justifiable for only a very narrow range of ‘Moneyness’. This result is due in part to the fact that the hedging instruments are very costly when their times-to-maturity are as high as 30 years, and in part to the fact that a time-to-maturity of 30 years is too long relative to the average time at which the annuity payments are made. Third, for the dynamic cash-flow hedges, the risk reduction is robust to changes in the times-to-maturity of the hedging instruments. This observation is in line with what was found by Zhou and Li (2017a), who argued that such robustness arises because in an annually adjusted hedge, the hedging instruments acquired at each time point are responsible for hedging the one-year-ahead uncertainty only. For similar reasons, we also observe that the results of the value hedges are robust to changes in the times-to-maturity of the hedging instruments.

**Figure 3.11** Performance of static cash-flow hedges (left panel), dynamic cash-flow hedges (middle panel), and static value hedges (right panel) at different degrees of ‘Moneyness’, when three alternative times-to-maturity, 10, 20 and 30 years, are considered
We now consider three alternative risk-free interest rates, 0.01, 0.03, and 0.04, while retaining all the other baseline assumptions. Note that according to equation (10), the hedgers’ risk-adjusted interest rate changes as we alter the assumed risk-free interest rate. The results based on the alternative risk-free interest rates are reported in Figure 3.12. It can be observed that the hedging performance of all the three types of hedges is highly robust to changes in the risk-free interest rate. The high robustness can be explained as follows:

- As the risk-free interest rate increases, the longevity deltas of both the annuity liability and hedging instruments become smaller in magnitude. As a consequence, the notional amounts \( h(i) \) in static hedges and \( h_t(i) \) in dynamic hedges, each of which is calculated as the ratio of the longevity deltas of the annuity liability and the relevant hedging instrument, remain fairly unchanged.

- As the risk-free interest rate increases, the risk-adjusted interest rate for discounting realised cash flows also increases. Because the cash flows associated with both the hedged and unhedged positions are discounted more heavily, the net effect tends to be small.
Figure 3.12 Performance of static cash-flow hedges (left panel), dynamic cash-flow hedges (middle panel), and static value hedges (right panel) at different degrees of ‘Moneyness’, when three alternative risk-free interest rates, 0.01, 0.03 and 0.04, are considered.
3.8 Further remarks

As parametric mortality indices are rich in information content and interpretable, they are likely to facilitate future development of the market for longevity risk transfers. This work complements the previous studies on parametric mortality indices in several aspects. First, we have introduced K-options written on the CBD mortality indices (which were found to satisfy the three key criteria for parametric mortality indices). We have provided explanations as to how K-calls and K-puts may be utilised by different types of hedgers who may have different exposures to K1 and K2 risks. We have also examined the relative performance between K-option hedges and K-forward hedges under different circumstances. Second, we have derived exact analytical pricing formulas for K-forwards and K-options, using the statistical and structural properties of the stochastic process in a risk-neutral version of the Cairns-Blake-Dowd model. The analytical pricing formulas not only enable us to more readily estimate the fair values of K-forwards and K-options in various market conditions, but also spare us from the need for computationally demanding nested simulations in the evaluation of dynamic cash-flow hedges and value hedges. Third, using K-forwards and K-options as hedging instruments, we have developed three types of longevity hedges: static cash-flow hedges, dynamic cash-flow hedges, and static value hedges. In all of them, the analytical expressions for the longevity deltas of K-forwards and K-options play an important role. We illustrate the three types of longevity hedges using real mortality data and previously estimated market prices of risk. Highlights of the empirical work include the following:

• For static / dynamic cash-flow hedges, a K-put hedge tends to yield a lower VaR compared to the corresponding K-forward hedge when the market prices of risk are high. However, if the market prices of risk are too high, both K-put and K-forward hedges may no longer be economically justifiable.

• For static cash-flow hedges, a K-put hedge tends to result in a smaller VaR compared to the corresponding K-forward hedge when the times-to-maturity of the hedging instruments are long.

• Value hedges are highly effective, even when the market prices of risk are high.

• Dynamic cash-flow hedges and value hedges are robust relative to changes in the times-to-maturity of the hedging instruments.

• All three types of hedges are robust relative to changes in the risk-free interest rate.
We acknowledge that European calls and puts are not the only options that can be written on the CBD mortality indices. Recently, Cairns and Boukfaoui (2017) considered a call spread on a hedge index (an index that is related to the liability being hedged), with a payoff function that is specified as follows:

\[
\max \left( 0, \min \left( \frac{HI - AP}{EP - AP}, 0 \right) \right)
\]  

(18)

where HI is the hedge index, and AP and EP represent the attachment and exhaustion points of the security respectively. In future research, it would be interesting to study K-options with a payoff function similar to (18). Such K-options may more easily attract capital market investors, as payoff functions similar to (18) are often seen in catastrophe bonds, which are widely traded in today’s market. In addition, we are aware that an annuity liability depends on the trajectories of the CBD mortality indices over a certain period of time but a European K-option is linked to the value of a CBD mortality index on the maturity date only. This mismatch may have resulted in some structural basis risk, which may be reduced if path-dependent K-options are used instead of European K-options.

We also acknowledge that a bivariate random walk with drift may not be the most appropriate process for the dynamics of the CBD mortality indices. For instance, using Tiao and Box’s (1981) procedure, Chan et al. (2014) found that a VARIMA(5,1,0) model is needed to adequately capture the serial- and cross-correlations in the CBD mortality indices. As a matter of fact, the pricing formulas presented in Section 3.4 are still valid when a VARIMA process is assumed for the CBD mortality indices instead, provided that \( E_t(Q_t[k_t]) \) and \( \text{Var}_t(Q_t[k_t]) \) in the pricing formulas are adapted accordingly. In more detail, suppose that under the real-world probability measure, the CBD mortality indices follow a VARIMA\((p,1,0)\) process

\[
\kappa_t - \kappa_{t-1} = \mu + \sum_{i=1}^{p} \Phi_i (\kappa_{t-i} - \kappa_{t-i-1}) + A z_t
\]  

(19)

where \( p \) is the autoregressive order, and \( \Phi_i, i = 1, 2, ..., p, \) are \( 2 \times 2 \) autoregressive coefficient matrices. As in Section 3.3, we assume further that the risk-neutral dynamics of the CBD mortality indices follow the same process as (19), except that the drift vector is changed to \( \tilde{\nu} = \mu - A \lambda \), where \( \lambda \) is the vector of market prices of risk. Under these model assumptions, it can be shown that

15 VARIMA stands for vector autoregressive integrated moving average.

16 The definitions of \( A, \mu, \) and \( z_t \) are provided in Section 3.3.
$E_t^{(Q)}[\kappa_T] = \bar{u} + C_1 E_t^{(Q)}[\kappa_{T-1}] + ... + C_{p+1} E_t^{(Q)}[\kappa_{T-(p+1)}]$

where

$C_i = I + \Phi_i \quad i = 1$

$C_i = \Phi_i - \Phi_{i-1} \quad i = 2, \ldots, p$

$C_i = -\Phi_p \quad i = p + 1$

$C_i = 0 \quad i = p + 2, \ldots$

$I$ is a $2\times2$ identity matrix and $0$ is a $2\times2$ zero matrix. Also, we have

$\text{Var}_t^{(Q)}[\kappa_T] = \sum_{i=1}^{T-t-1} \Psi_i (AA') \Psi_i$

where $\Psi_i = \sum_{j=1}^{i} \Psi_{i-j} C_{j}$ and $\Psi_0 = I$. Then, for $i = 1, 2$, $E_t^{(Q)}[\kappa^{(i)}_T]$ and $\text{Var}_t^{(Q)}[\kappa^{(i)}_T]$ in the pricing formulas can be obtained as the $i$th element in $E_t^{(Q)}[\kappa_T]$ and the $(i, i)$th element in $\text{Var}_t^{(Q)}[\kappa_T]$, respectively. Analytical pricing formulas may also be obtained when the CBD mortality indices are assumed to follow other processes, but the derivation may be more involved.

As Chan et al. (2014) argued, when developing parametric mortality indices, we need a model that possesses the new-data-invariant property (a property that ensures that the resulting indices are tractable). It has been found that among all of the candidate models considered by Dowd et al. (2010), the Cairns-Blake-Dowd model is the only one that possesses this crucial property. Admittedly, the Cairns-Blake-Dowd model may not provide the best goodness-of-fit to historical data, and consequently, a longevity hedge that is developed from the CBD mortality indices is subject to some residual risk arising from the age, period, and/or cohort effects that the Cairns-Blake-Dowd model does not capture. The residual risk may possibly be modelled and incorporated into the evaluation of hedge effectiveness using an age-period-cohort structure that is fitted to the residuals of the estimated Cairns-Blake-Dowd model. This important analysis is left for further research.
3.9 Derivation of formulas

Here we consider a (European) K-call on the ith CBD mortality index, with issue date $t_0$, maturity date $T$, and strike value $K$. We use $r_f$ to represent the risk-free interest rate, $I_A$ to denote an indicator function which equals 1 if event $A$ occurs and 0 otherwise, and the following shorthand notation:

$$B = \frac{K - E_t^{(Q)}[\kappa_T^{(i)}] }{\sqrt{Var_t^{(Q)}[\kappa_T^{(i)}]}}$$

We also use the fact that, for $t < T$,

$$\frac{\partial B}{\partial \kappa_t^{(i)}} = \frac{-1}{\sqrt{Var_t^{(Q)}[\kappa_T^{(i)}]}}$$

which follows from equations (5) and (6). The price of the K-call at time $t$ for $t_0 \leq t < T$ is given by

$$C_t^{(i)}(T, K) = E_t^{(Q)} \left[ (1 + r_f)^{-T-t} \max(\kappa_T^{(i)} - K, 0) \right]$$

Aside, we have

$$E_t^{(Q)}[\kappa_T^{(i)}I_{(\kappa_T^{(i)}>K)}] = \int_{\kappa_T^{(i)}=K}^{\infty} \frac{x}{\sqrt{2\pi Var_t^{(Q)}[\kappa_T^{(i)}]}} \exp(-\frac{(x-E_t^{(Q)}[\kappa_T^{(i)}])^2}{2Var_t^{(Q)}[\kappa_T^{(i)}]}) \, dx$$

$$= E_t^{(Q)}[\kappa_T^{(i)}](1 - \Phi(B)) - \frac{1}{\sqrt{2\pi Var_t^{(Q)}[\kappa_T^{(i)}]}} \int_{-\infty}^{B} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} u^2\right) \exp\left(-\frac{1}{2} u^2\right) \, du$$

$$= E_t^{(Q)}[\kappa_T^{(i)}](1 - \Phi(B)) + \frac{1}{\sqrt{2\pi Var_t^{(Q)}[\kappa_T^{(i)}]}} \phi(B).$$

It follows that

$$C_t^{(i)}(T, K) = (1 + r_f)^{-T-t} \sqrt{Var_t^{(Q)}[\kappa_T^{(i)}]} (\phi(B) - B(1 - \Phi(B)))$$
For $t_0 \leq t < T$, the time-$t$ value of the longevity delta for the $K$-call is given by

$$\Delta_t^{(i,C)}(T, K) = \frac{\partial C_t^{(i)}(T, K)}{\partial \kappa_t^{(i)}}$$

$$= (1 + r_f)^{-(T-t)} \sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]} \left( \frac{\partial \phi(B)}{\partial \kappa_t^{(i)}} - \frac{\partial B}{\partial \kappa_t^{(i)}} \Phi(B) + \frac{\partial B \Phi(B)}{\partial \kappa_t^{(i)}} \right)$$

$$= (1 + r_f)^{-(T-t)} \sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]} \left( \frac{B \phi(B) - (-1) + (-\Phi(B)) + (-B \Phi(B))}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \right)$$

$$= (1 + r_f)^{-(T-t)} (1 - \Phi(B)).$$

For $t_0 \leq t < T$, the time-$t$ value of the longevity gamma for the $K$-call is given by

$$\Gamma_t^{(i,C)}(T, K) = \frac{\partial}{\partial \kappa_t^{(i)}} \Delta_t^{(i,C)}(T, K)$$

$$= -(1 + r_f)^{-(T-t)} \frac{\partial}{\partial \kappa_t^{(i)}} \Phi(B)$$

$$= (1 + r_f)^{-(T-t)} \frac{1}{\sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]}} \phi(B).$$

Now we consider a (European) $K$-put on the $i$th CBD mortality index, with issue date $t_0$, maturity date $T$, and strike value $K$. The price of the $K$-put at time $t$ for $t_0 \leq t < T$ is given by

$$P_t^{(i)}(T, K) = E_t^{(Q)} \left[ (1 + r_f)^{-(T-t)} \max(K - \kappa_t^{(i)}, 0) \right]$$

$$= (1 + r_f)^{-(T-t)} E_t^{(Q)} \left[ (K - \kappa_t^{(i)}) I_{\{\kappa_t^{(i)} < K\}} \right]$$

$$= (1 + r_f)^{-(T-t)} \left( KE_t^{(Q)} \left[ I_{\{\kappa_t^{(i)} < K\}} \right] - E_t^{(Q)} \left[ \kappa_t^{(i)} I_{\{\kappa_t^{(i)} < K\}} \right] \right)$$

$$= (1 + r_f)^{-(T-t)} \left( K \Phi(B) - E_t^{(Q)} \left[ \kappa_t^{(i)} I_{\{\kappa_t^{(i)} < K\}} \right] \right).$$

Aside, using the results from above, we have

$$E_t^{(Q)} \left[ \kappa_t^{(i)} I_{\{\kappa_T^{(i)} < K\}} \right] = E_t^{(Q)} \left[ \kappa_t^{(i)} \right] - E_t^{(Q)} \left[ \kappa_t^{(i)} I_{\{\kappa_T^{(i)} > K\}} \right]$$

$$= E_t^{(Q)} \left[ \kappa_T^{(i)} \right] - E_t^{(Q)} \left[ \kappa_T^{(i)} \Phi(B) \right] - \sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]} \phi(B)$$

$$= E_t^{(Q)} \left[ \kappa_T^{(i)} \Phi(B) - \sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]} \phi(B) \right].$$

It follows that

$$P_t^{(i)}(T, K) = (1 + r_f)^{-(T-t)} \left( \sqrt{\text{Var}_t^{(Q)}[\kappa_T^{(i)}]} \phi(B) + (K - E_t^{(Q)}[\kappa_T^{(i)}]) \Phi(B) \right).$$
For \( t_0 \leq t < T \), the time-\( t \) value of the longevity delta for the K-put is given by

\[
\Delta^{(i,P)}_t(T, K) = \frac{\partial P^{(i)}_t(T, K)}{\partial \kappa^{(i)}_t} = (1 + r_f)^{-(T-t)} \sqrt{\text{Var}^{(Q)}_t[\kappa^{(i)}_T]} \left( \frac{\partial \phi(B)}{\partial \kappa^{(i)}_t} + \frac{\partial B \Phi(B)}{\partial \kappa^{(i)}_t} \right)
\]

\[
= (1 + r_f)^{-(T-t)} \sqrt{\text{Var}^{(Q)}_t[\kappa^{(i)}_T]} \left( \frac{\partial \phi(B)}{\partial \kappa^{(i)}_t} + \frac{\partial B}{\partial \kappa^{(i)}_t} \Phi(B) + \frac{\partial \Phi(B)}{\partial \kappa^{(i)}_t} B \right)
\]

\[
= (1 + r_f)^{-(T-t)} \sqrt{\text{Var}^{(Q)}_t[\kappa^{(i)}_T]} \left( B \phi(B) + (-\Phi(B)) + (-B \phi(B)) \right)
\]

\[
= -(1 + r_f)^{-(T-t)} \Phi(B).
\]

For \( t_0 \leq t < T \), the time-\( t \) value of the longevity gamma for the K-put is given by

\[
\Gamma^{(i,P)}_t(T, K) = \frac{\partial}{\partial \kappa^{(i)}_t} \Delta^{(i,P)}_t(T, K)
\]

\[
= -(1 + r_f)^{-(T-t)} \frac{\partial}{\partial \kappa^{(i)}_t} \Phi(B)
\]

\[
= (1 + r_f)^{-(T-t)} \frac{1}{\sqrt{\text{Var}^{(Q)}_t[\kappa^{(i)}_T]}} \phi(B)
\]
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